Derivation of Parseval's theorem

$$\hat{\vec{E}}(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}(\vec{r},t)\, e^{i\omega t}\, d\omega \qquad (1)$$

$$\int_{-\infty}^{\infty} |\vec{E}(\vec{r},t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\vec{E}}(\vec{r},\omega)\cdot\hat{\vec{E}}^*(\vec{r},\tilde{\omega}) \underbrace{\int_{-\infty}^{\infty} e^{i(\omega-\tilde{\omega})t}\, dt}_{2\pi\,\delta(\omega-\hat{\omega})}\, d\hat{\omega}\, d\omega$$

$$= \int_{-\infty}^{\infty} \hat{\vec{E}}(\vec{r},\omega)\cdot\int_{-\infty}^{\infty} \vec{E}^*(\vec{r},\hat{\omega})\,\delta(\omega-\tilde{\omega})\, d\hat{\omega}\, d\omega$$

$$= \int_{-\infty}^{\infty} \hat{\vec{E}}(\vec{r},\omega)\,\hat{\vec{E}}^*(\vec{r},\omega)\, d\omega = \int_{-\infty}^{\infty} |\hat{\vec{E}}(\vec{r},\omega)|^2 d\omega$$

since $\vec{E}(\vec{r},-\omega) = \vec{E}^*(\vec{r},\omega)$:

$$\boxed{\int_{-\infty}^{\infty} |\vec{E}(\vec{r},t)|^2 dt = 2\int_{0}^{\infty} |\hat{\vec{E}}(\vec{r},\omega)|^2 d\omega} \qquad (2)$$

$\longrightarrow$ provided $\hat{E}(\vec{r},\omega)$ is defined as in Eq. (1)