Solution of Maxwell's Equations in Free Space

$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t) \longrightarrow \nabla \times \nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\nabla \times \vec{B}(\vec{r},t) \tag{1}$$

$$\nabla \times \vec{B}(\vec{r},t) = \mu_0 \vec{j}(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) \tag{2}$$

$$(1) + (2) : \left[\nabla^2 - \frac{\partial^2}{\partial t^2}\frac{1}{c^2}\right]\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\nabla\rho(\vec{r},t) + \mu_0 \frac{\partial}{\partial t}\vec{j}(\vec{r},t) \tag{3}$$

Free space:

$$\boxed{\left[\nabla^2 - \frac{\partial^2}{\partial t^2}\frac{1}{c^2}\right]\vec{E}(\vec{r},t) = 0} \tag{4}$$

$$\rightarrow \left[\nabla^2 - \frac{\partial^2}{\partial t^2}\frac{1}{c^2}\right]E_x(\vec{r},t) = 0$$

$$\rightarrow \left[\nabla^2 - \frac{\partial^2}{\partial t^2}\frac{1}{c^2}\right]E_y(\vec{r},t) = 0$$

$$\rightarrow \left[\nabla^2 - \frac{\partial^2}{\partial t^2}\frac{1}{c^2}\right]E_z(\vec{r},t) = 0$$

E_x, E_y, E_z not independent $\longrightarrow$ $\boxed{\nabla \cdot \vec{E} = 0}$ (5)

Solution of $\left[\nabla^2 - \frac{\partial^2}{\partial t^2}\frac{1}{c^2}\right]f(\vec{r},t) = 0$: (6)

① If $f(\vec{r},t) = f(x,t)$ (one-dimensional problem)

$$\longrightarrow \text{d'Alembert} : \boxed{f(x,t) = f(t \pm x/c)} \tag{7}$$

Proof: $$\frac{\partial^2}{\partial x^2} f(t \pm x/c) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(t \pm x/c) = 0$$

$$\frac{\partial^2}{\partial x^2} f(t \pm x/c) = \frac{\partial}{\partial x}\left[\frac{\partial f(s)}{\partial s}\underbrace{\frac{\partial (t \pm x/c)}{\partial x}}_{\pm 1/c}\right] = \pm\frac{1}{c}\left[\frac{\partial^2 f(s)}{\partial s^2}\underbrace{\frac{\partial (t \pm x/c)}{\partial x}}_{\pm \frac{1}{c}}\right]$$

$$\frac{\partial^2}{\partial t^2} f(t \pm x/c) = \frac{\partial}{\partial t}\left[\frac{\partial f(s)}{\partial s}\underbrace{\frac{\partial (t \pm x/c)}{\partial t}}_{1}\right] = \frac{\partial^2 f(s)}{\partial s^2}$$

qed

② Separation of Variables:

$f(\vec{r},t) = R(\vec{r})\,T(t)$ $\longrightarrow$ insert in Eq. (6):

$$\underbrace{c^2 \frac{1}{R}\nabla^2 R}_{-\text{const.}} - \underbrace{\frac{1}{T}\frac{\partial^2}{\partial t^2} T}_{-\text{const.}} = 0$$

Thus: $$\frac{\partial^2}{\partial t^2} T(t) + \text{const.}\, T(t) = 0 \qquad (8)$$

$$\nabla^2 R(\vec{r}) + \frac{\text{const.}}{c^2} R(\vec{r}) = 0 \qquad (9)$$

const. = arbitrary constant : Set const. $= \omega^2$

$$\boxed{\begin{aligned} \frac{\partial^2}{\partial t^2} T(t) + \omega^2 T(t) &= 0 \qquad (10) \\ \nabla^2 R(\vec{r}) + \frac{\omega^2}{c^2} R(\vec{r}) &= 0 \qquad (11) \end{aligned}}$$

$R(\vec{r}), T(t)$ = real functions!

Solution of (10): harmonic differential equation

$$T(t) = A_\omega \cos \omega t + B_\omega \sin \omega t \qquad (12)$$

$$= \operatorname{Re}\left\{ c_\omega e^{-i\omega t} \right\}$$

Thus: $$f_\omega(\vec{r},t) = \underset{\uparrow \text{real!}}{R(\vec{r},\omega)} \left[A_\omega \cos \omega t + B_\omega \sin \omega t \right] \qquad (13)$$

$$\boxed{f_\omega(\vec{r},t) = \operatorname{Re}\left\{ \underbrace{c_\omega R(\vec{r},\omega)}_{\substack{\text{complex} \\ \text{amplitude } \underline{R}(\vec{r},\omega)}} e^{-i\omega t} \right\}} \qquad (14)$$

The general solution is the superposition of all homogeneous solutions. Each ω gives a homogeneous solution. Therefore:

ω = discrete, e.g. cavity:

$$\boxed{f(\vec{r},t) = \sum_n f_{\omega_n}(\vec{r},t) = \sum_n \operatorname{Re}\left\{ \underline{R}(\vec{r},\omega_n) e^{-i\omega_n t} \right\}} \qquad (15)$$

ω = continuous, e.g. free space:

$$f(\vec{r},t) = \int^{\infty} \operatorname{Re}\left\{ \underline{\tilde{R}}(\vec{r},\omega) e^{-i\omega t} \right\} d\omega \qquad (16)$$