Solutions of Midterm Exam

Problem (1): 50 points

1.1. one-dimensional solution of waveequation (homework 1)

$$f(z,t) = f(t - z/c)$$

thus: $$\vec{E}(z,t) = E_0 \cos\left[\omega_0 (t - z/c)\right] \frac{\sin\left[\Delta\omega (t - z/c)\right]}{\Delta\omega (t - z/c)} \vec{n}_x \quad (7)$$

Conditions to be satisfied: $$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E}(z,t) = 0 \quad (2)$$

$$\nabla \cdot \vec{E}(z,t) = 0 \quad (1)$$

1.2. $$\hat{\vec{E}}(z,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(z,t)\, e^{i\omega t}\, dt \quad (1)$$

$$= \frac{2E_0}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{i\omega_0 (t - z/c) + i\omega t} \frac{\sin[\Delta\omega (t - z/c)]}{\Delta\omega (t - z/c)}\, dt \right. \quad (1)$$

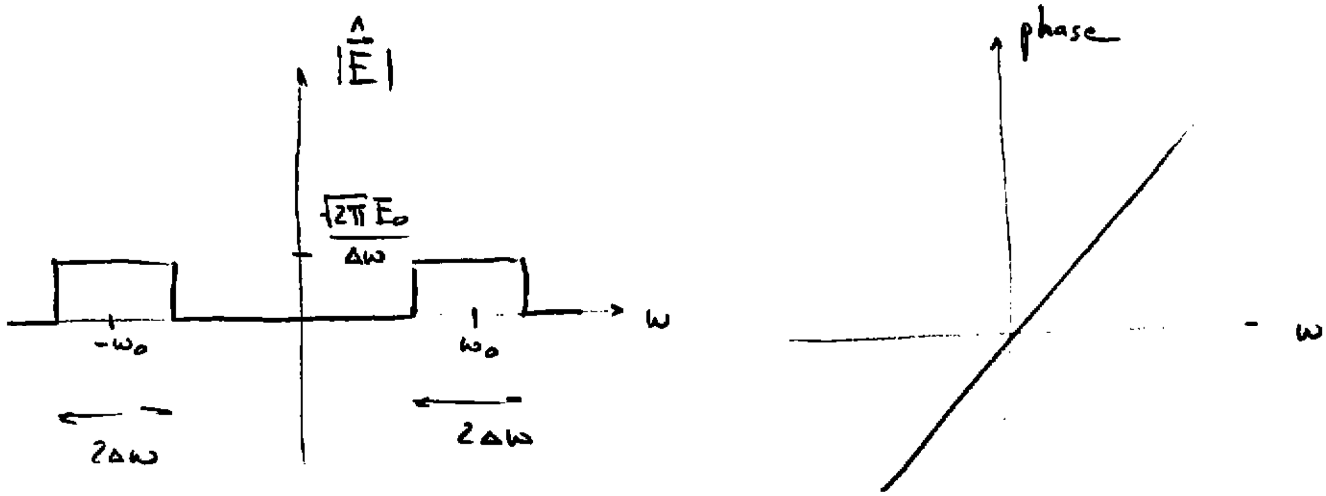
$$\left. + \int_{-\infty}^{\infty} e^{-i\omega_0 (t - z/c) + i\omega t} \frac{\sin[\Delta\omega (t - z/c)]}{\Delta\omega (t - z/c)}\, dt \right] \vec{n}_x$$

$$= \frac{2E_0}{\sqrt{2\pi}} \vec{n}_x\, e^{i\omega z/c} \left[\int_{-\infty}^{\infty} e^{i(\omega+\omega_0)\hat{t}} \frac{\sin \Delta\omega \hat{t}}{\Delta\omega \hat{t}}\, d\hat{t} + \int_{-\infty}^{\infty} e^{i(\omega-\omega_0)\hat{t}} \frac{\sin \Delta\omega \hat{t}}{\Delta\omega \hat{t}}\, d\hat{t} \right] \quad (2)$$

using hint:

$$\hat{\vec{E}}(z,\omega) = \sqrt{\tfrac{\pi}{2}}\,\frac{E_o}{\Delta\omega}\,\vec{n}_x\; e^{i\omega\frac{z}{c}}\Big[\mathrm{sign}(\omega_o+\Delta\omega+\omega) - \mathrm{sign}(\omega_o-\Delta\omega+\omega) + \mathrm{sign}(-\omega_o+\Delta\omega+\omega) - \mathrm{sign}(-\omega_o-\Delta\omega+\omega)\Big] \qquad (2)$$

Sketch:



1.3. solution for $\hat{\vec{E}}$ in vacuum:

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\hat{\vec{E}} = 0 \qquad (2)$$

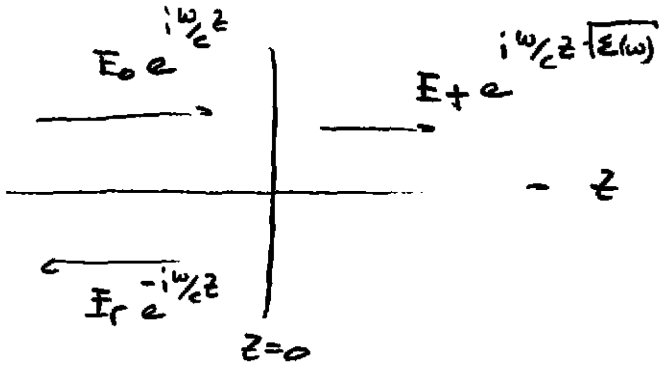
solution for $\hat{\vec{E}}$ in dispersive medium:

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\varepsilon(\omega)\right]\hat{\vec{E}} = 0 \qquad (3)$$

→ pulse in dispersive medium:

$$\hat{\vec{E}}(z,\omega) = \sqrt{\tfrac{\pi}{2}}\,\frac{E_o}{\Delta\omega}\,\vec{n}_x\; e^{i\frac{\omega}{c}\sqrt{\varepsilon(\omega)}\,z}\Big[\mathrm{sign}(\ldots)\ \ldots\Big]$$

1.4 boundary conditions:



E_0, E_r, E_t = spectral amplitudes

(1) $E_0(z=0) + E_r(z=0) = E_t(z=0)$

(2) $H_0(z=0) + H_r(z=0) = H_t(z=0)$ (2)

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(1) $E_0 + E_r = E_t$

(2) $E_0 - E_r = \sqrt{\varepsilon(\omega)}\, E_t$ (4)

$$r(\omega) = \frac{E_r}{E_0} = \frac{1-\sqrt{\varepsilon(\omega)}}{1+\sqrt{\varepsilon(\omega)}}$$

(4)

$$t(\omega) = \frac{E_t}{E_0} = \frac{2}{1+\sqrt{\varepsilon(\omega)}}$$

1.5. Spectrum of transmitted field :

$$\hat{\vec{E}}_+(z,\omega) = \sqrt{\frac{\pi}{2}}\,\frac{E_0}{\Delta\omega}\,t(\omega)\,e^{i\frac{\omega}{c}\sqrt{\varepsilon(\omega)}\,z}\,\left[\mathrm{sign}(..)\,...\right]\vec{n}_x \qquad 5$$

backtransform :

$$\vec{E}_+(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\vec{E}}_+(z,\omega)\,e^{-i\omega t}\,d\omega \qquad 2$$

$$= \frac{E_0}{2\Delta\omega}\int_{-\infty}^{\infty} t(\omega)\,e^{-i\omega\left[t-\frac{\sqrt{\varepsilon(\omega)}}{c}z\right]}\left[\mathrm{sign}(..)\,...\right]d\omega\;\vec{n}_x \qquad 3$$

$$= \frac{E_0}{\Delta\omega}\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} t(\omega)\,e^{-i\omega\left[t-\sqrt{\varepsilon(\omega)}/c\,z\right]}\,d\omega\;\vec{n}_x \qquad 3$$

$$+\frac{E_0}{\Delta\omega}\int_{-\omega_0-\Delta\omega}^{-\omega_0+\Delta\omega} t(\omega)\,e^{-i\omega\left[t-\sqrt{\varepsilon(\omega)}/c\,z\right]}\,d\omega\;\vec{n}_x$$

since $\hat{\vec{E}}_+(z,-\omega) = \hat{\vec{E}}_+^{*}(z,\omega)$:

$$\vec{E}_+(z,t) = \frac{2E_0}{\Delta\omega}\,\mathrm{Re}\left\{\int_{\omega_0-\Delta\omega}^{\omega_0+\Delta\omega} t(\omega)\,e^{-i\omega\left[t-z/c\sqrt{\varepsilon(\omega)}\right]}\,d\omega\right\} \qquad 2$$

Problem 2 : 30 points

2.1 $$E = E_{rad} + E_{kin}$$ 2

E_{rad}: radiated energy during acceleration process

E_{kin}: final kinetic energy

$$E_{kin} = m_0 c^2 \left[\left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} - 1 \right]$$ 3

$$E_{rad} = \int_0^{t_0} P(t)\, dt \qquad \text{where} \quad t_0 = v_0/a_0 = \Delta s / v_0$$ 2

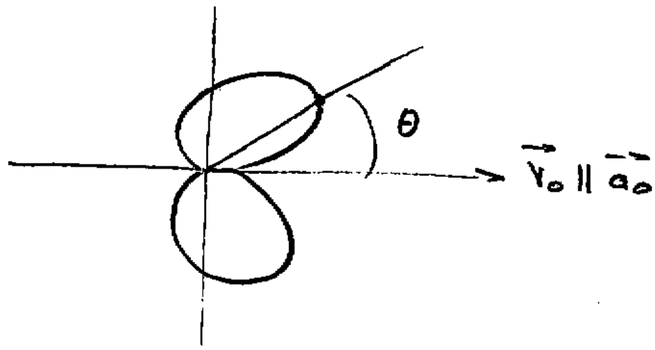
$$P(t) = \frac{1}{4\pi\varepsilon_0} \frac{2 q^2 a_0^2}{3c^3} \left[1 - a_0^2 t^2/c^2\right]^{-3}$$

$$(a_0 = v_0^2/\Delta s)$$

$$E_{rad} = \frac{1}{4\pi\varepsilon_0} \frac{2q^2 a_0^2}{3c^3} \int_0^{\Delta s/v_0} \frac{1}{\left[1 - a_0^2 t^2/c^2\right]^3}\, dt$$ 3

$$= \frac{\Delta s/v_0}{4\left(1 - v_0^2/c^2\right)^2} + \frac{3\, \Delta s/v_0}{8\left(1 - v_0^2/c^2\right)} + \frac{3 c \Delta s}{8 v_0^2} \operatorname{arctanh}\left(\frac{v_0}{c}\right)$$

2.2



$$\frac{dE}{d\Omega} = \int_0^{t_0 = v_0/a_0 = \Delta S/v_0} \frac{dP}{d\Omega}(t)\, dt$$ 5

forward radiated energy:

$$E^{\uparrow} = \int_0^{\pi/2}\int_0^{2\pi} \frac{dE}{d\Omega} \sin\theta \, d\phi \, d\theta$$

5

backward radiated energy:

$$E^{\downarrow} = \int_{\pi/2}^{\pi}\int_0^{2\pi} \frac{dE}{d\Omega} \sin\theta \, d\phi \, d\theta$$

for $\vec{a}_0 \parallel \vec{v}_0$: $$\frac{dP}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^2}{4\pi}\right) \frac{|\hat{R}(\hat{R}\cdot\vec{a}_0) - \vec{a}_0|^2}{c^3 (1 - \hat{R}\cdot\vec{a}_0 t/c)^5}$$

$$\frac{dE}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^2}{4\pi}\right)\frac{1}{c^3}\left|\hat{R}(\hat{R}\cdot\vec{a}_0) - \vec{a}_0\right|^2 \int_0^{t_0} \frac{1}{[1 - \hat{R}\cdot\vec{a}_0 t/c]^5}\, dt$$

express $dP/d\Omega$ and $dE/d\Omega$ in terms of θ, ϕ :

$$\hat{R} \cdot \frac{\vec{a}_o}{a_o} = \cos\theta$$

$$\left| \hat{R}\,(\hat{R}\cdot\vec{a}_o) - \vec{a}_o \right|^2 = a_o^2 \left[\hat{R}\cos\theta - \frac{\vec{a}_o}{a_o} \right] \cdot \left[\hat{R}\cos\theta - \frac{\vec{a}_o}{a_o} \right]$$

$$= a_o^2 \left[\cos^2\theta\, \hat{R}\cdot\hat{R} + \frac{\vec{a}_o}{a_o}\cdot\frac{\vec{a}_o}{a_o} - 2\cos\theta\, \hat{R}\cdot\frac{\vec{a}_o}{a_o} \right]$$

$$= a_o^2 \left[\cos^2\theta + 1 - 2\cos^2\theta \right] = a_o^2 \sin^2\theta$$

$$\frac{dE}{d\Omega} = \frac{1}{4\pi\varepsilon_o} \left(\frac{q^2}{4\pi} \right) \frac{a_o^2}{c^3} \int_0^{t_o} \frac{\sin^2\theta}{\left[1 - \frac{a_o}{c} t \cos\theta \right]^5}\, dt \qquad (5)$$

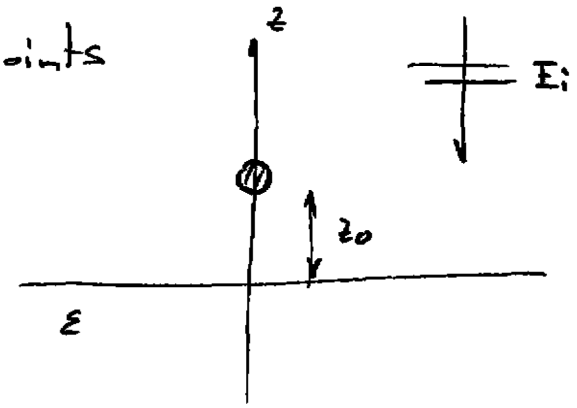
thus:
$$\frac{E^\uparrow}{E^\downarrow} = \frac{\int_0^{\pi/2} \int_0^{t_o} \sin^3\theta \left[1 - \frac{a_o}{c} t \cos\theta \right]^{-5} dt\, d\theta}{\int_{\pi/2}^{\pi} \int_0^{t_o} \sin^3\theta \left[1 - \frac{a_o}{c} t \cos\theta \right]^{-5} dt\, d\theta} \qquad (5)$$

with time-dependence integrated:

$$\frac{E^\uparrow}{E^\downarrow} = \frac{\int_0^{\pi/2} \sin^3\theta \cos^{-1}\theta \left[1 - \left(1 - \frac{v_o}{c}\cos\theta \right)^{-4} \right] d\theta}{\int_{\pi/2}^{\pi} \sin^3\theta \cos^{-1}\theta \left[1 - \left(1 - \frac{v_o}{c}\cos\theta \right)^{-4} \right] d\theta}$$

Problem 3 : 40 points

3.1



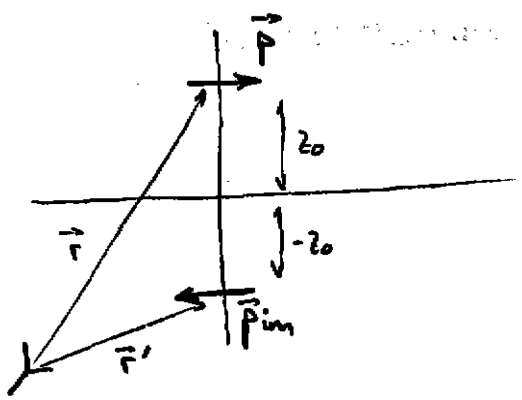
$$\vec{E}_i = \mathrm{Re}\left\{ E_i\, e^{-ikz}\, e^{-i\omega t} \right\} \qquad k = 2\pi/\lambda$$ 1

$$\vec{E}_r = \mathrm{Re}\left\{ E_i\, \frac{\varepsilon-1}{\varepsilon+1}\, e^{ikz}\, e^{-i\omega t} \right\}$$ 2

$$\vec{E}_o(z,t) = \mathrm{Re}\left\{ \underbrace{E_i \left(e^{-ikz_0} + \frac{\varepsilon-1}{\varepsilon+1}\, e^{ikz_0} \right)}_{\vec{E}_o(z_0)} e^{-i\omega t} \right\}$$ 2

3.2 $$\alpha = 4\pi\varepsilon_0\, a^3\, \frac{\varepsilon_p - 1}{\varepsilon_p + 2}$$ (see homework 5) 5

3.3



$$\vec{p} = \alpha\left(\vec{E}_o + \vec{E}_{int}\right)$$ 3

$$\vec{E}_{int}(\vec{r}) = \omega^2 \mu_0\, \overleftrightarrow{G}(\vec{r},\vec{r}')\, \vec{p}_{im}$$

$$= -\omega^2 \mu_0 \left(\frac{\varepsilon-1}{\varepsilon+1}\right) \overleftrightarrow{G}(z_0,-z_0)\, \vec{p}$$ 3

Combining: $$\vec{p} = \alpha \vec{E}_0 - \omega^2 \mu_0 \alpha \left(\frac{\varepsilon-1}{\varepsilon+1}\right) \overleftrightarrow{G}(z_0,-z_0)\vec{p} \qquad (2)$$

$$\vec{p} = \underbrace{\alpha \left[\overleftrightarrow{I} + \omega^2 \mu_0 \alpha \left(\frac{\varepsilon-1}{\varepsilon+1}\right) \overleftrightarrow{G}(z_0,-z_0)\right]^{-1}}_{\overleftrightarrow{\alpha}_{eff}} \vec{p} \qquad (3)$$

$$\overleftrightarrow{G}(\vec{r},\vec{r}') \approx \overleftrightarrow{G}_{NF}(\vec{r},\vec{r}') = \frac{e^{ikR}}{4\pi R} \frac{1}{k^2R^2}\left[-\overleftrightarrow{I} + \frac{3\vec{R}\vec{R}}{R^2}\right] \qquad (2)$$

with $\vec{R} = \vec{r} - \vec{r}' = 2z_0\, \vec{n}_z$

$R = 2z_0$

$$\rightarrow \overleftrightarrow{G}(z_0,-z_0) = \frac{e^{2ikz_0}}{8\pi z_0} \frac{1}{4k^2z_0^2}\left[-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + 3\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\right] \qquad (5)$$

$$\rightarrow \overleftrightarrow{\alpha}_{eff} = \alpha \begin{bmatrix} 1-\omega^2\mu_0\alpha\left(\frac{\varepsilon-1}{\varepsilon+1}\right)\frac{e^{2ikz_0}}{32\pi k^2 z_0^3} & 0 & 0 \\ 0 & 1-\omega^2\mu_0\alpha\left(\frac{\varepsilon-1}{\varepsilon+1}\right)\frac{e^{2ikz_0}}{32\pi k^2 z_0^3} & 0 \\ 0 & 0 & 1+2\omega^2\mu_0\alpha\left(\frac{\varepsilon-1}{\varepsilon+1}\right)\frac{e^{2ikz_0}}{32\pi k^2 z_0^3} \end{bmatrix}^{-1}$$

invert: $$\overleftrightarrow{\alpha}_{eff} = \alpha \begin{bmatrix} \frac{1}{1-\omega^2\mu_0\alpha\ldots} & 0 & 0 \\ 0 & \frac{1}{1-\omega^2\mu_0\alpha\ldots} & 0 \\ 0 & 0 & \frac{1}{1+2\omega^2\mu_0\alpha\ldots} \end{bmatrix}$$

since $\vec{E}_0 = E_0 \vec{n}_x$ (no y and z components) :

$$\overleftrightarrow{\alpha}_{eff} = \alpha_{eff} = \frac{\alpha}{1 - \omega^2 \mu_0 \alpha \left(\frac{\varepsilon-1}{\varepsilon+1}\right) \frac{\exp[2ikz_0]}{32\pi k^2 z_0^3}} \qquad (5)$$

3.4. $P_{scattered} = P_{particle} + P_{interface} + P_{interference}$

→ interference term cancels upon integration over $\Omega = 4\pi$

thus : $$P_{sc} = \frac{\omega^4}{12\pi\varepsilon_0 c^3}\left[p^2 + p_{im}^2\right] = \frac{\omega^4}{12\pi\varepsilon_0 c^3}\left[1 + \left(\frac{\varepsilon-1}{\varepsilon+1}\right)^2\right] p^2 \qquad (2)$$

use $\vec{p} = \alpha_{eff} \vec{E}_0(z=z_0)$

$$P_{sc} = \frac{\omega^4 \alpha_{eff}^2}{12\pi\varepsilon_0 c^3}\left[\frac{2(\varepsilon^2+1)}{(\varepsilon+1)^2}\right] E_0^2(z=z_0) \qquad (3)$$