FINAL EXAM SOLUTIONS AND GRADING

1.1 Class notes $\rightarrow$ $\overline{P} = \dfrac{p_o^2}{4\pi\varepsilon_o} \dfrac{\omega_o^4}{3c^3} e^{-\gamma(t - r/c)}$ $\boxed{5}$

$$\overline{E} = \int_{r/c}^{\infty} \overline{P}\, dt = \frac{1}{\gamma} \overline{P}$$ $\boxed{5}$

Also possible: $\overline{E} = \displaystyle\int_{\Omega}\int_{\omega} \frac{dW}{d\Omega d\omega} d\omega\, d\Omega$ or $\boxed{3}$

$$\overline{E} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{dW}{d\Omega d\omega} d\omega \sin\theta\, d\theta\, d\phi$$ or $\boxed{2}$

where $\dfrac{dW}{d\Omega d\omega}$ from class notes

1.2 $\overline{E} = \dfrac{p_o^2}{4\pi\varepsilon_o} \dfrac{\omega_o^3}{3c^3} \dfrac{1}{Q}$ where $Q = \gamma/\omega_o$ $\boxed{2}$

$\rightarrow$ more $\overline{E}$ if Q small ? $\rightarrow$ No

Substitute $\gamma = \dfrac{e^2\omega_o^2}{6\pi\varepsilon_o mc^3}$ $\rightarrow$ $\underbrace{\overline{E} = \frac{1}{2} m\omega_o^2 |\vec{r}_o|^2}_{\text{initial energy}}$ $\boxed{3}$

The oscillator always radiates its entire initial energy no matter how big γ is !

2.1

$$E_x(x,y,z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \hat{E}_x(k_x,k_y;z=0)\, e^{i[k_x x + k_y y + k_z z]}\, dk_x dk_y \quad (2)$$

for $z>0$ and $k_z = \sqrt{k^2-k_x^2-k_y^2}$

where $\hat{E}_x(k_x,k_y;z=0) = \frac{E_0}{2\pi}\iint_{-\infty}^{\infty} E_x(x,y;z=0)\, e^{-i[k_x x + k_y y]}\, dx\, dy$

$$= \frac{E_0}{2} W_0^2\, e^{-(k_x^2+k_y^2)\frac{W_0^2}{4}} \quad (2)$$

thus: $E_x(x,y,z) = \frac{E_0 W_0^2}{4\pi}\iint_{-\infty}^{\infty} e^{-(k_x^2+k_y^2)\frac{W_0^2}{4}}\, e^{i[k_x x + k_y y + k_z z]}\, dk_x dk_y \quad (1)$

for $z>0$ and $k_z = \sqrt{k^2-k_x^2-k_y^2}$ $\left(k^2 = \frac{\omega^2}{c^2}\right)$

2.2

$\nabla\cdot\vec{E} = 0$ where $\vec{E} = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \begin{bmatrix} \hat{E}_x(k_x,k_y;z=0) \\ \hat{E}_y(k_x,k_y;z=0) \\ \hat{E}_z(k_x,k_y;z=0) \end{bmatrix} e^{i[k_x x + k_y y + k_z z]}\, dk_x dk_y \quad (3)$

$\rightarrow E_y = 0 \Rightarrow \hat{E}_y = 0$

$\rightarrow \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial z}E_z = 0 \longrightarrow \hat{E}_z(k_x,k_y;0) = -\frac{k_x}{k_z}\hat{E}_x(k_x,k_y;0) \quad (2)$

thus: $E_z(x,y,z) = \frac{E_0 W_0^2}{4\pi}\iint_{-\infty}^{\infty}\left(-\frac{k_x}{k_z}\right) e^{-(k_x^2+k_y^2)\frac{W_0^2}{4}}\, e^{i[k_x x + k_y y + k_z z]}\, dk_x dk_y \quad (5)$

2.3 $\hat{E}_x = \dfrac{E_o w_o^2}{2} e^{-(s_x^2+s_y^2)\frac{k^2 w_o^2}{4}}$ where $s_x = \frac{x}{r}$, $s_y = \frac{y}{r}$, $s_z = \frac{z}{r}$ (3)

(2)

$$\hat{E}_z = -\frac{E_o w_o^2}{2}\left(\frac{s_x}{s_z}\right) e^{-(s_x^2+s_y^2)\frac{k^2 w_o^2}{4}}$$

Farfields: $E_x^\infty = -ik s_z \dfrac{e^{ikr}}{r} \hat{E}_x(s_x, s_y, s_z)$

$$= -ik \frac{E_o w_o^2}{2} \frac{e^{ikr}}{r} s_z e^{-(s_x^2+s_y^2)\frac{k^2 w_o^2}{4}}$$ (3)

$$E_z^\infty = -ik s_z \frac{e^{ikr}}{r} \hat{E}_z(s_x, s_y, s_z)$$

$$= +ik \frac{E_o w_o^2}{2} \frac{e^{ikr}}{r} s_x e^{-(s_x^2+s_y^2)\frac{k^2 w_o^2}{4}}$$

Spherical coordinates: $s_x = \cos\phi \sin\theta$

$s_y = \sin\phi \sin\theta$

$s_z = \cos\theta$

$$\vec{E}^\infty(r,\theta,\phi) = -ik\frac{E_o w_o^2}{2} e^{-\frac{k^2 w_o^2}{4}\sin^2\theta}\left[\cos\theta\, \vec{n}_x - \sin\theta\cos\phi\, \vec{n}_z\right]\frac{e^{ikr}}{r}$$ (2)

spherical wave $\rightarrow$ 1.) spherical phase fronts: e^{ikr} ok.

2.) $1/r$ farfield decay ok. (3)

3.) $\vec{E}^\infty \cdot \vec{n}_r = 0$ ($\vec{n}_r$ = radial unit vector)

$\vec{n}_r = \cos\phi\sin\theta\, \vec{n}_x + \sin\phi\sin\theta\, \vec{n}_y + \cos\theta\, \vec{n}_z$

$$\longrightarrow \begin{bmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta\cos\phi \end{bmatrix} = 0$$ ok.

3.1 d'Alembert : $f(z) \rightarrow f(z-ct)$ $\boxed{2}$

$$\vec{E}(\vec{r},t) = \vec{E}_0 \frac{1}{1+\left(\frac{z-ct}{z_0}\right)^2} \cos\left[\frac{\omega_0}{c}(z-ct)\right] \qquad \boxed{3}$$

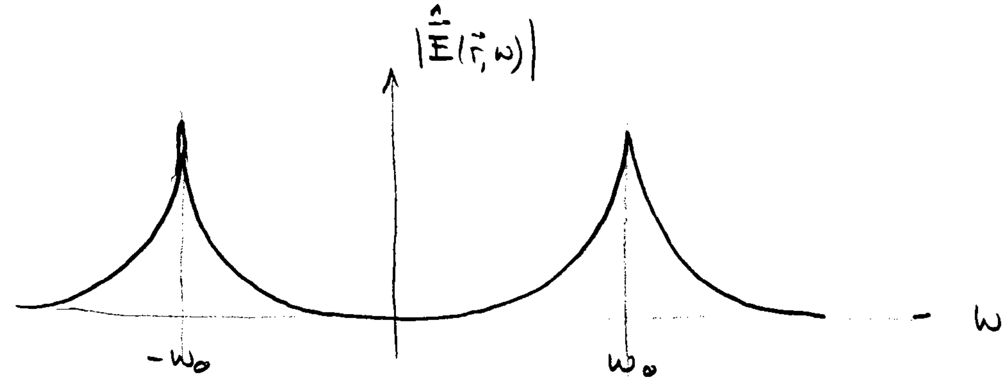
3.2 Spectrum: $$\hat{\vec{E}}(\vec{r},\omega) = \frac{\vec{E}_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\cos\left(\frac{\omega_0}{c}z - \omega_0 t\right)}{1+(z-ct)^2/z_0^2} e^{i\omega t}\, dt \qquad \boxed{2}$$

Substitution: $x = \frac{\omega_0}{c} z - \omega_0 t$

$$\hat{\vec{E}}(\vec{r},\omega) = \frac{\vec{E}_0}{\sqrt{2\pi}} \frac{1}{\omega_0} \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2/\left(\frac{\omega_0}{c} z_0\right)^2} e^{-i\frac{\omega}{\omega_0}x}\, dx\; e^{i\frac{\omega}{c}z} \qquad \boxed{2}$$

Integrate: $$\hat{\vec{E}}(\vec{r},\omega) = \sqrt{\frac{\pi}{2}} \frac{z_0}{2c} \vec{E}_0 e^{i\frac{\omega}{c}z} \left[e^{-\frac{z_0}{c}|\omega-\omega_0|} + e^{-\frac{z_0}{c}|\omega+\omega_0|} \right] \qquad \boxed{4}$$

Sketch :



$\boxed{2}$

3.3 $$I(\vec{r},t) = \vec{S}(\vec{r},t)\cdot\vec{n}_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} |\vec{E}(\vec{r},t)|^2$$

$$\frac{dW}{dA} = \int_{-\infty}^{\infty} I(\vec{r},t)\, dt = \sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{-\infty}^{\infty} |\vec{E}(\vec{r},t)|^2 dt \underset{\text{Parseval}}{=} 2\sqrt{\frac{\varepsilon_0}{\mu_0}} \int_0^{\infty} |\hat{\vec{E}}(\vec{r},\omega)|^2 d\omega \qquad \boxed{2}$$

$$\frac{dW}{dA\,d\omega} = 2\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\left|\hat{\vec{E}}(\vec{r},\omega)\right|^2$$

$$= |\vec{E}_0|^2 \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{\pi z_0^2}{4c^2}\left[e^{-\frac{z_0}{c}|\omega-\omega_0|} + e^{-\frac{z_0}{c}|\omega+\omega_0|}\right]^2 \qquad \boxed{3}$$

3.4 $$\frac{dW}{dA} = \int_0^\infty \frac{dW}{dA\,d\omega}\,d\omega = 2\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_0^\infty \left|\vec{E}(\vec{r},\omega)\right|^2 d\omega \qquad \boxed{3}$$

integrate: $$\frac{dW}{dA} = |\vec{E}_0|^2\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{\pi z_0}{4c}\left[1 + \left(1 + 2\frac{z_0 \omega_0}{c}\right)e^{-2\frac{z_0\omega_0}{c}}\right] \qquad \boxed{7}$$

3.5 Field in dispersive medium: $\left[\nabla^2 + \frac{\omega^2}{c^2}n^2(\omega)\right]\hat{\vec{E}}(z,\omega) = 0$ $\boxed{2}$

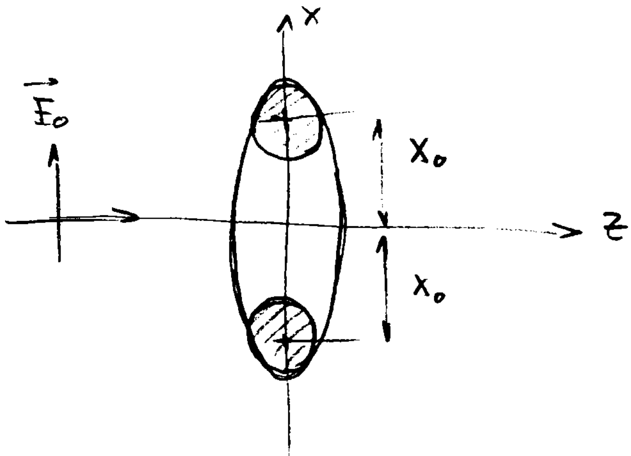
$$\vec{E}(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\vec{E}}(z,\omega)\,e^{-i\omega t}\,d\omega$$

where

$$\hat{\vec{E}}(\vec{r},\omega) = \sqrt{\frac{\pi}{2}}\,\frac{z_0}{2c}\,\vec{E}_0\,e^{i\frac{\omega}{c}n(\omega)z}\left[e^{-\frac{z_0}{c}|\omega-\omega_0|} + e^{-\frac{z_0}{c}|\omega+\omega_0|}\right]$$

$\boxed{3}$

4.1



$$\vec{E}(\vec{r}) = \vec{E}_o(\vec{r}) + \omega^2\mu_o \sum_{n=1}^{N} \overleftrightarrow{G}(\vec{r},\vec{r}_n)\,\vec{p}_n \qquad \text{where } \vec{p}_n = \alpha_n \vec{E}_n(\vec{r}_n) \qquad (2)$$

2 particles:
$$\vec{E}(\vec{r}_1) = \vec{E}_o(\vec{r}_1) + \omega^2\mu_o\alpha\,\overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\,\vec{E}(\vec{r}_2) \qquad (3)$$
$$\vec{E}(\vec{r}_2) = \vec{E}_o(\vec{r}_2) + \omega^2\mu_o\alpha\,\overleftrightarrow{G}(\vec{r}_2,\vec{r}_1)\,\vec{E}(\vec{r}_1)$$

solve for $\vec{E}(\vec{r}_1)$:
$$\vec{E}(\vec{r}_1) = \left[\overleftrightarrow{I} - \omega^4\mu_o^2\alpha^2\,\overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\,\overleftrightarrow{G}(\vec{r}_2,\vec{r}_1)\right]^{-1}\left[\vec{E}_o(\vec{r}_1) + \omega^2\mu_o\alpha\,\overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\,\vec{E}_o(\vec{r}_2)\right]$$

similarly for $\vec{E}(\vec{r}_2)$

dipole moments:
$$\vec{p}_1 = \alpha\left[\overleftrightarrow{I} - \omega^4\mu_o^2\alpha^2\,\overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\,\overleftrightarrow{G}(\vec{r}_2,\vec{r}_1)\right]^{-1}\left[\vec{E}_o(\vec{r}_1) + \omega^2\mu_o\alpha\,\overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\,\vec{E}_o(\vec{r}_2)\right] \qquad (5)$$
$$\vec{p}_2 = \alpha\left[\overleftrightarrow{I} - \omega^4\mu_o^2\alpha^2\,\overleftrightarrow{G}(\vec{r}_2,\vec{r}_1)\,\overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\right]^{-1}\left[\vec{E}_o(\vec{r}_2) + \omega^2\mu_o\alpha\,\overleftrightarrow{G}(\vec{r}_2,\vec{r}_1)\,\vec{E}_o(\vec{r}_1)\right]$$

Special case: $\vec{E}_o = E_o\,\vec{n}_x \;\rightarrow\; \vec{E}_o(\vec{r}_1) = \vec{E}_o(\vec{r}_2) \;\rightarrow\; \vec{p}_1 = \vec{p}_2 = p\,\vec{n}_x$

near-field:
$$\overleftrightarrow{G}(\vec{r}_1,\vec{r}_2) = \frac{e^{2ikx_o}}{8\pi x_o}\,\frac{1}{4k^2x_o^2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \overleftrightarrow{G}(\vec{r}_2,\vec{r}_1) \qquad (1)$$

no retardation: $\exp[2ikx_o] \approx 1$

$$\overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\,\overleftrightarrow{G}(\vec{r}_2,\vec{r}_1) = \frac{1}{(8\pi x_o)^2(4k^2x_o^2)^2}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

$$\left[\overleftrightarrow{I} - \omega^4 \mu_0^2 \alpha^2 \overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\overleftrightarrow{G}(\vec{r}_2,\vec{r}_1)\right]^{-1} =$$ 1

$$\begin{bmatrix} (1-4A^2)^{-1} & 0 & 0 \\ 0 & (1-A^2)^{-1} & 0 \\ 0 & 0 & (1-A^2)^{-1} \end{bmatrix} \quad \text{where} \quad A^2 = \frac{\alpha^2}{(32\pi\varepsilon_0 x_0^3)^2}$$

$$\left[\vec{E}_0(\vec{r}_1) + \omega^2\mu_0\alpha \overleftrightarrow{G}(\vec{r}_1,\vec{r}_2)\vec{E}_0(\vec{r}_2)\right] = E_0 \vec{n}_x \left[1 + 2A\right]$$ 1

together: $$\vec{p}_1 = \frac{\alpha}{(1-4A^2)} E_0 \left[1+2A\right] \vec{n}_x = \frac{\alpha}{(1-2A)} \vec{E}_0$$

$$\vec{p}_2 = \frac{\alpha}{(1-4A^2)} E_0 \left[1+2A\right] \vec{n}_x = \frac{\alpha}{(1-2A)} \vec{E}_0$$ 1

4.2 $$\vec{p}_{ell} = \vec{p}_1 + \vec{p}_2 = \frac{2\alpha}{(1-2A)} \vec{E}_0$$ 2

thus: $$\overleftrightarrow{\alpha}_{eff} = \frac{2\alpha}{1 - \alpha/(16\pi\varepsilon_0 x_0^3)}$$ 8

5. s-pol. $\frac{E_r^{(s)}}{E_i} = \frac{k_{z_1} - k_{z_2}}{k_{z_1} + k_{z_2}} = r^s \qquad k_{z_1} = n k_0 \sin\theta$ 5

p-pol $\frac{E_r^{(p)}}{E_i} = \frac{k_{z_1} - n^2 k_{z_2}}{k_{z_1} + n^2 k_{z_2}} = r^p \qquad k_{z_1} = n k_0 \sin\theta$

$k_{x_1} = k_{x_2}$: $\quad k_0^2 - k_{z_2}^2 = n^2 k_0^2 - n^2 k_0^2 \sin^2\theta$

$$k_{z_2} = k_0 \sqrt{1 - n^2 \cos^2\theta}$$

total internal reflection: $\quad k_{z2} = i\alpha(\theta)$ with $\alpha = k_0 \sqrt{n^2\cos^2\theta - 1}$ 5

$k_{z_1} = \beta(\theta)$ with $\beta = n k_0 \sin\theta$

$$r^s(\theta) = \frac{\beta - i\alpha}{\beta + i\alpha} = \frac{(\beta - i\alpha)^2}{\beta^2 - \alpha^2} \longrightarrow \varphi^s(\theta) = \arctan\left(\frac{-2\alpha\beta}{\beta^2 - \alpha^2}\right)$$

5

$$r^p(\theta) = \frac{\beta - i\alpha n^2}{\beta + i\alpha n^2} = \frac{(\beta - i\alpha n^2)^2}{\beta^2 - \alpha^2 n^4} \longrightarrow \varphi^p(\theta) = \arctan\left(\frac{-2\alpha\beta n^2}{\beta^2 - \alpha^2 n^4}\right)$$

For each reflection: $\Delta\varphi = |\varphi^s - \varphi^p| = \pi/4$

where $\varphi^s(\theta) = -\arctan\left(\frac{2n\sin\theta\sqrt{n^2\cos^2\theta - 1}}{[1 + n^2\sin^2\theta - n^2\cos^2\theta]}\right)$

$\varphi^p(\theta) = -\arctan\left(\frac{2n^3\sin\theta\sqrt{n^2\cos^2\theta - 1}}{[n^4 + n^2\sin^2\theta - n^6\cos^2\theta]}\right)$

solve numerically for θ 5