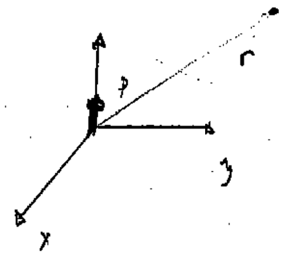

$$\vec{E}(\vec{r},t) = \frac{\sin\theta}{4\pi\epsilon_0}\frac{1}{c^2 r}\frac{d^2}{dt^2}p_z\left(t-\frac{r}{c}\right)\hat{e}_\theta \qquad [1]$$

$$\left\{\frac{d^2}{dt^2}+\gamma_0\frac{d}{dt}+\omega_0^2\right\}p_z(t) = 0$$

solving the above equation

$$\lambda^2 + \gamma_0\lambda + \omega_0^2 = 0$$

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma_0^2 - 4\omega_0^2}}{2}$$

assuming $4\omega_0^2 > \gamma_0^2$ $\quad \lambda = \frac{\gamma_0}{2} \pm i\Omega_0$

where $\Omega_0^2 = \frac{4\omega_0^2 - \gamma_0^2}{4}$ {shift with respect to natural frequency}

then

$$p_z(t) = p_0 e^{-\frac{\gamma_0 t}{2}}\cos\left[\Omega_0 t + \phi_0\right] \qquad [2]$$

where p_0 and ϕ_0 are constant

From eq [1] and eq [2]

since

$$\tau = t - \frac{|\vec{r} - \vec{r}_e(\tau)|}{c}$$

since $|\vec{r}| \gg |\vec{r}_e(\tau)|$ then τ

$$\tau \approx t - \frac{r}{c}$$

therefore

$$\frac{d^2 p(t-\frac{r}{c})}{dt^2} \approx p_0 e^{-\frac{\gamma_0}{2}(t-\frac{r}{c})}\left\{\left[-\Omega_0^2 + \frac{\gamma_0^2}{4}\right]\cos\left[\Omega_0\left(t-\frac{r}{c}\right)+\phi_0\right] - \frac{\gamma_0\Omega_0}{2}\sin\left[\Omega_0\left(t-\frac{r}{c}\right)+\phi_0\right]\right\}$$

since $\Omega_0^2 \gg \gamma_0^2$ and $\Omega_0^2 \gg \gamma_0\Omega_0$ then

$$\frac{d^2}{dt^2}p\left[t-\frac{r}{c}\right] \approx -p_0\Omega_0^2 e^{-\gamma_0(t-\frac{r}{c})}\cos\left[\Omega_0\left(t-\frac{r}{c}\right)+\phi_0\right]$$

therefore

$$\boxed{\vec{E}(\vec{r},t) = \frac{-\sin\theta}{4\pi\epsilon_0}\frac{\Omega_0^2}{c^2 r}p_0 e^{-\frac{\gamma_0}{2}(t-\frac{r}{c})}\cos\left[\Omega_0\left(t+\frac{r}{c}\right)+\phi_0\right]} \qquad [3]$$

$$\bar{P} = \frac{1}{T}\int d\Omega \int_{-T/2}^{T/2} dt \,\{ r^2 E^2(r,t)\,\epsilon_0 c \}$$ {average power radiated}

T is the period of the cosine function

$$\bar{P} = \frac{1}{T}\int_{-T/2}^{T/2} dt \int_0^{\pi}\int_0^{2\pi} d\phi\, d\theta \, \frac{\sin^3\theta\, \epsilon_0 c\, p_0^2}{[4\pi\epsilon_0]^2 c^4 r^2}\,\Omega_0^4 r^2 \cos^2[\Omega_0(t-r/c)+\phi_0]\, e^{-\gamma_0(t-r/c)}$$

$$\bar{P} = \frac{1}{T}\int_{-T/2}^{T/2} dt\, \frac{2 p_0^2 \Omega_0^4}{4^2\pi\epsilon_0 c^3}\int_0^{\pi} \sin^3\theta\, d\theta \, \cos^2[\Omega_0(t-r/c)+\phi_0]\, e^{-\gamma_0(t-r/c)}$$

$$P = \frac{1}{4\pi\epsilon_0}\,\frac{p_0^2\Omega_0^4}{3c^3}\,\frac{2}{T}\int_{-T/2}^{T/2} \cos^2[\Omega_0(t-r/c)+\phi_0]\, e^{-\gamma_0(t-r/c)}\, dt$$

since $T \ll \frac{1}{\gamma_0}$ then

$$P \approx \frac{1}{4\pi\epsilon_0}\,\frac{p_0^2\Omega_0^4}{3c^3}\, e^{-\gamma_0(t-r/c)} \cdot \frac{2}{T}\int_{-T/2}^{T/2} \cos^2[\Omega_0(t-r/c)+\phi_0]\, dt$$

$$\boxed{P \approx \frac{1}{4\pi\epsilon_0}\,\frac{p_0^2\Omega_0^4}{3c^3}\, e^{-\gamma_0(t-r/c)}}$$

2 This is a very different approach to find the equation for an oscillator. Actually this a problem of the Book of Jackson

$$m\frac{dv}{dt} - \underbrace{m\tau\frac{d^2v}{dt^2}}_{\text{radiative force}} = F_{ext}(t)$$

where $\tau = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2}{3c^3 m}$

By using Laplace transforms $t \to s$

$$m\dot{V}(s) - m\tau s\dot{V}(s) = F_{ext}(s)$$

$$m\dot{V}(s) = \frac{F_{ext}(s)}{1-\tau s} = -\frac{1}{\tau}\frac{F_{ext}(s)}{s-\frac{1}{\tau}}$$

by inverse transformation

$$m\dot{v}(t) = -\frac{1}{\tau}\,\theta(t)\,e^{1/\tau s} \circledast \left[F(t)\right]$$

where $\theta(t)$ is the step function

$$m\dot{v}(t) = \int_0^\infty \frac{-e^{+q/\tau}}{\tau} F_{ex}(t-q)\, dq$$

let $\frac{-q}{\tau} = s$

$$\boxed{m\dot{v}(t) = \int_0^\infty e^{-s} F_{ext}(t+\tau s)\, ds}$$

Integral equation that includes the radiation effects

then if τ is small we expand the above equation in Taylor series

$$m\dot{v}(t) = \int_0^\infty e^{-s} \sum_{n=0}^\infty \frac{d^n F(t)_{ex}}{dt^n}\frac{(\tau s)^n}{n!}\, ds$$

$$m\dot{v}(t) = \sum_{n=0}^\infty \frac{d^n F(t)_{ex}}{dt^n}\frac{\tau^n}{n!}\int_0^\infty e^{-s} s^n\, ds$$

$$m\dot{v}(t) = \sum_{n=0}^\infty \frac{d^n F_{ext}(t)}{dt^n}\tau^n\frac{\Gamma(n+1)}{n!}$$

$$\boxed{m\dot{v}(t) = \sum_{n=0}^\infty \frac{d^n F_{ext}(t)}{dt^n}\tau^n}$$

For an oscillator $F_{ext} = -m\omega_0^2 x(t)$

Keeping the first terms of the expansion

$$m\dot{v}(t) = -m\omega_0^2 x(t) - m\omega_0^2 \dot{x}(t)\tau$$

then

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2\tau\frac{dx}{dt} + \omega_0^2 x = 0}$$

then

$$x = A e^{-\frac{\gamma_0}{2}t} \cos[\Omega_0 t + \phi_0]$$

where $\gamma_0 = \omega_0^2 \tau$

$$\Omega_0^2 = \omega_0^2 - \frac{\gamma_0^2}{4}$$

then

$$a(t) = A e^{-\frac{\gamma_0}{2}t} \left[\left(-\Omega_0^2 + \frac{\gamma_0^2}{4}\right)\cos[\Omega_0 t + \phi_0] + \gamma_0 \Omega_0 \sin[\Omega_0 t + \phi_0]\right]$$

since $\Omega_0^2 > \gamma_0^2 > \gamma_0 \Omega_0$

then

$$a(t) \approx -A e^{-\frac{\gamma_0}{2}t} \Omega_0^2 \cos[\Omega_0 t + \phi_0]$$

thus the acceleration is damped by a decay constant $\frac{\gamma_0}{2}$

3

$$\hat{E}(r,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E(t,r)\, e^{i\omega t}\, dt$$

$$\hat{E}(r,\omega) = \frac{E_0(r)}{\sqrt{2\pi}} \int_{r/c}^{\infty} e^{-\frac{\gamma_0}{2}\left(t-\frac{r}{c}\right)} \cos\left[\omega_0\left(t-\frac{r}{c}\right)+\phi_0\right] e^{i\omega t}\, dt$$

where $E_0(r) = \dfrac{-\sin\theta}{4\pi\epsilon_0}\dfrac{\Omega_0^2}{c^2 r}\, p_0$

Let $\tau = t - r/c$

$$\hat{E}(r,\omega) = \frac{E_0(r)}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{\gamma_0}{2}\tau} \cos[\omega_0\tau+\phi_0]\, e^{i\omega\left[\tau+\frac{r}{c}\right]}\, d\tau$$

$$\hat{E}(r,\omega) = \frac{E_0(r)}{2\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{\gamma_0\tau}{2}}\left(e^{i[\omega_0\tau+\phi_0]} + e^{-i[\omega_0\tau+\phi_0]}\right) e^{i\omega\left[\tau+\frac{r}{c}\right]}\, d\tau$$

$$\hat{E}(r,\omega) = \frac{E_0(r)}{2\sqrt{2\pi}} e^{i\frac{\omega r}{c}} \int_0^{\infty}\left[e^{i[\omega+\omega_0]\tau + i\phi_0 - \frac{\gamma_0}{2}\tau} + e^{i[\omega-\omega_0]\tau - i\phi_0 - \frac{\gamma_0}{2}\tau}\right] d\tau$$

$$\hat{E}(r,\omega) = \frac{E_0(r)}{2\sqrt{2\pi}} e^{i\frac{\omega r}{c}} \left[-\frac{e^{i\phi_0}}{i(\omega+\omega_0)-\frac{\gamma_0}{2}} - \frac{e^{-i\phi_0}}{+i[\omega-\omega_0]-\frac{\gamma_0}{2}}\right]$$

$$\hat{E}(r,\omega) = \frac{-E_0(r)}{2\sqrt{2\pi}} e^{i\frac{\omega r}{c}} \left[\frac{e^{i\phi_0}}{i(\omega+\omega_0)-\frac{\gamma_0}{2}} - \frac{e^{-i\phi_0}}{i[\omega-\omega_0]-\frac{\gamma_0}{2}}\right]$$

then

$$|E(r,\omega)|^2 = \frac{E_0^2(r)}{4(2\pi)}\left[\frac{1}{(\omega+\omega_0)^2+\frac{\gamma_0^2}{4}} + \frac{1}{(\omega-\omega_0)^2+\frac{\gamma_0^2}{4}} + 2\,\mathrm{Re}\left\{\frac{e^{2i\phi_0}}{\left[i(\omega+\omega_0)-\frac{\gamma_0}{2}\right]\left[-i[\omega-\omega_0]-\frac{\gamma_0}{2}\right]}\right\}\right]$$

$$|E(r,\omega)|^2 = \frac{E_0^2(r)}{8\pi}\left[\frac{1}{(\omega+\omega_0)^2+\frac{\gamma_0^2}{4}} + \frac{1}{(\omega-\omega_0)^2+\frac{\gamma_0^2}{4}} + 2\,\mathrm{Re}\left\{\frac{e^{2i\phi}}{\frac{\gamma_0^2}{4}+(\omega+\omega_0)(\omega-\omega_0) - i\left[\omega+\omega_0-\omega+\omega_0\right]\frac{\gamma_0}{2}}\right\}\right]$$

$$|E(r,\omega)|^2 = \frac{E_0^2(r)}{8\pi}\left[\frac{1}{(\omega+\omega_0)^2+\frac{\gamma_0^2}{4}} + \frac{1}{(\omega-\omega_0)^2+\frac{\gamma_0^2}{4}} + 2\,\mathrm{Re}\left\{\frac{e^{i2\phi_0}}{\frac{\gamma_0^2}{4}+(\omega+\omega_0)(\omega-\omega_0) - i\omega_0\gamma_0}\right\}\right]$$

then

$$W = 2\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^2 \epsilon_0 c \left|\hat{E}(r,\omega)\right|^2 \sin\theta \, d\theta \, d\phi \, d\omega$$

$$W = \frac{1}{4\pi}\left\{\frac{2\pi \epsilon_0 \omega_0^4 \epsilon_0 c \cancel{r^2} p_0^2}{(4\pi\epsilon_0)^2 c^4 \cancel{r^2}}\int \sin^3\theta \, d\theta \left[\frac{1}{(\omega+\omega_0)^2 + \frac{\gamma_0^2}{4}} + \frac{1}{(\omega-\omega_0)^2 + \frac{\gamma_0^2}{4}}\right.\right.$$

$$\left.\left. + 2\,\mathrm{Re}\left\{\frac{e^{-i2\phi_0}}{\frac{\gamma_0^2}{4} + (\omega+\omega_0)(\omega-\omega_0) - i\omega_0\gamma_0}\right\}\right]\right\} d\omega$$

$$W = \frac{1}{4\pi}\cdot\frac{2\, p_0^2 \omega_0^4}{(4\pi\epsilon_0) 3 c^3}\int_0^{\infty} \frac{1}{(\omega+\omega_0)^2 + \frac{\gamma_0^2}{4}} + \frac{1}{(\omega-\omega_0)^2 + \frac{\gamma_0^2}{4}}$$

$$+ 2\,\mathrm{Re}\left\{\frac{e^{-i2\phi_0}}{\frac{\gamma_0^2}{4} + (\omega+\omega_0)(\omega-\omega_0) - i\omega_0\gamma}\right\} d\omega$$

since $\omega_0 \gg \gamma_0$ then

$$W \approx \frac{1}{4\pi}\frac{2}{4\pi\epsilon_0}\frac{p_0^2\omega_0^4}{3c^3}\int_{-\infty}^{\infty}\frac{d\omega}{(\omega-\omega_0)^2 + \frac{\gamma_0^2}{4}}$$

$$W \propto \cancel{\frac{1}{4\pi}}\frac{\cancel{2}}{4\pi\epsilon_0}\frac{p_0^2\omega_0^4}{3c^3}\frac{\cancel{2\pi}}{\gamma_0}$$

$$\boxed{W \approx \frac{1}{4\pi\epsilon_0}\frac{p_0^2\omega_0^4}{3c^3\gamma_0}}$$

at $t=0$ the total energy of the oscilator is

$$E = \frac{1}{2} m\omega_0^2 x_0^2 = \frac{1}{2}\frac{m\omega_0^2 e^2 x_0}{e^2} = \frac{p_0^2}{2e^2} m\omega_0^2$$

then

$$\frac{W}{E} = \frac{\frac{1}{4\pi\epsilon_0}\frac{p_0^2\omega_0^4}{3c^3\gamma_0}}{\frac{p_0^2}{2e^2} m\omega_0^2} = \frac{1}{\gamma_0}\frac{1}{4\pi\epsilon_0}\frac{\omega_0^2 e^2}{c^3 m}\cdot\frac{2}{3}$$

$$\frac{W}{E} = \frac{1}{\gamma_0}\,\omega_0^2\,\underbrace{\frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2}{c^3 m}} = \frac{1}{\gamma_0}\omega_0^2\tau$$

Where τ is the proportionality term of the radiative force, and since

$\gamma_0 \equiv \omega_0^2\tau$

then

$$\boxed{\frac{W}{E} = 1}$$

$$\hbar\omega_0 = \frac{1}{2} m\omega_0^2 x_0^2$$

$$x_0 = \sqrt{\frac{2\hbar}{\omega_0 m}} = \sqrt{\frac{2\hbar\lambda}{2\pi c m}}$$

$$\boxed{x_0 = \sqrt{\frac{\hbar\lambda_0}{\pi c m}} = \frac{1}{\pi}\sqrt{\frac{h\lambda_0}{2cm}}}$$

$\lambda_0 = 500\ nm$ $\quad$ $m = 9.11\times10^{-31}\ kg$

$$\boxed{x_0 = 2.478\ \mathring{A}}$$

1]

$$\vec{E}_x(t) = E_0 \cos \omega_0 t \, e^{-(\sigma_0 t)^2} \hat{n}_x$$

$$\vec{E}(z,t) = E_0 \cos\left[\omega_0\left[\frac{z}{c} - t\right]\right] e^{-\left[\sigma_0^2\left(\frac{z}{c} - t\right)^2\right]} \hat{n}_x$$

$$\vec{E}(z,t) = E_0 \cos\left[\omega_0\left[\frac{z}{c} - t\right]\right] e^{-\sigma_0^2\left[\frac{z}{c} - t\right]^2}$$

$$\frac{dW}{dA} = \int_{-\infty}^{\infty} dt \, E_0^2 c \epsilon_0 \left| \cos\left[\omega_0\left(\frac{z}{c} - t\right)\right] e^{-\sigma_0^2\left(\frac{z}{c} - t\right)^2} \right|^2$$

then

$$\frac{dW}{dA} = E_0^2 c \epsilon_0 \int_{-\infty}^{\infty} \cos^2\left[k_0 z - \omega_0 t\right] e^{-2\sigma_0^2\left(\frac{z}{c} - t\right)^2} dt$$

we apply the parseval theorem

$$E(\omega) = \frac{E_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos\left[\left[\frac{z}{c} - t\right]\omega_0\right] e^{-\sigma_0^2\left(\frac{z}{c} - t\right)^2} e^{i\omega t} dt$$

Let $u = \frac{z}{c} - t$

$$E(\omega) = \frac{E_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos\left[u\omega_0\right] e^{-\sigma_0^2 u^2} e^{i\frac{z\omega}{c}} e^{-i\omega u} du$$

$$E(\omega) = \frac{E_0}{\sqrt{2\pi}} e^{i\frac{z\omega}{c}} \int_{-\infty}^{\infty} \frac{e^{i\omega_0 u} + e^{-i\omega_0 u}}{2} e^{-\sigma_0^2 u^2} e^{-i\omega u} du$$

$$E(\omega) = \frac{E_0}{2\sqrt{2\pi}} e^{i\frac{z\omega}{c}} \left\{ \int_{-\infty}^{\infty} e^{i(\omega_0 - \omega)u} e^{-\sigma_0^2 u^2} du + \int_{-\infty}^{\infty} e^{-i(\omega_0 + \omega)u} e^{-\sigma_0^2 u^2} du \right]$$

$$E(\omega) = \frac{E_0}{2\sqrt{2\pi}} e^{i\frac{z\omega}{c}} \left\{ \left[\int_{-\infty}^{\infty} e^{-\sigma_0^2\left[u^2 - \frac{(\omega_0 - \omega)i}{\sigma_0^2} + \frac{1}{4\sigma_0^4}(\omega_0 - \omega)^2 i^2 - \frac{1}{4\sigma_0^4}(\omega_0 - \omega)^2 i^2\right]} du \right.\right.$$

$$+ \int_{-\infty}^{\infty} e^{-\sigma_0^2\left[u^2 + \frac{i(\omega_0 + \omega)}{\sigma_0^2} + \frac{1}{4\sigma_0^4}(\omega_0 + \omega)^2 i^2 - \frac{1}{4\sigma_0^4}(\omega_0 + \omega)^2 i^2\right]} du$$

$$E(\omega) = \frac{E_0}{2\sqrt{2\pi}} e^{i\frac{z\omega}{c}} \left\{ e^{-\frac{1}{4\sigma_0^2}(\omega_0 - \omega)^2} \int_{-\infty}^{\infty} e^{-\sigma_0^2\left[u - \frac{1}{2\sigma_0^2}(\omega_0 - \omega)i\right]^2} du \right.$$

$$\left. e^{-\frac{1}{4\sigma_0^2}(\omega_0 + \omega)^2} \int_{-\infty}^{\infty} e^{-\sigma_0^2\left[u + \frac{1}{2\sigma_0^2}(\omega_0 + \omega)i\right]^2} du \right\}$$

$$E(\omega) = \frac{E_0}{2\sqrt{2\pi}} e^{i\frac{z\omega}{c}} \frac{\sqrt{\pi}}{\sigma_0} \left[e^{-\frac{1}{4\sigma_0^2}(\omega_0 - \omega)^2} + e^{-\frac{1}{4\sigma_0^2}(\omega_0 + \omega)^2} \right]$$

then $$|E(\omega)|^2 = \frac{E_0^2}{\gamma_0^2}\left[e^{-\frac{1}{4\gamma_0^2}(\omega_0-\omega)^2} + e^{-\frac{1}{4\gamma_0^2}(\omega_0+\omega)^2} \right]^2$$

$$|E(\omega)|^2 = \frac{E_0^2}{8\gamma_0^2}\left[e^{-\frac{1}{2\gamma_0^2}(\omega_0-\omega)^2} + e^{-\frac{1}{2\gamma_0^2}(\omega_0+\omega)^2} + 2e^{-\frac{1}{4\gamma_0^2}\left[(\omega_0-\omega)^2+(\omega_0+\omega)^2\right]} \right]$$

therefore

$$\frac{dW}{dA} = 2\int_0^\infty c\epsilon_0 \, |E(\omega)|^2 \, d\omega$$

since $\gamma_0 < \omega_0$

then

$$\frac{dW}{dA} \approx 2\int_{-\infty}^{\infty} e^{-\frac{1}{2\gamma_0^2}(\omega_0-\omega)^2} \frac{E_0^2}{8\gamma_0} c\epsilon_0$$

$$\frac{dW}{dA} \approx \frac{1}{4\gamma_0^2} E_0^2 c\epsilon_0 \int_{-\infty}^{\infty} e^{-\frac{1}{2\gamma_0^2}(\omega_0-\omega)^2} d\omega$$

$$\boxed{\frac{dW}{dA} \approx \frac{\sqrt{2\pi}}{4}\frac{\gamma_0}{\gamma_0^2} E_0^2 c\epsilon_0 = \frac{\sqrt{\pi}\, E_0^2 c\epsilon_0}{2\sqrt{2}\,\gamma_0}}$$

$$\frac{dW}{dA\,d\omega} = 2\,|E(\omega)|^2 c\epsilon_0 \qquad \omega > 0$$

$$\frac{dW}{dA\,d\omega} \approx \frac{2c\epsilon_0 E_0^2}{8\gamma_0^2} e^{-\frac{1}{2\gamma_0^2}(\omega_0-\omega)^2} \qquad \omega > 0$$

$$\boxed{\frac{dW}{dA\,d\omega} \approx \frac{1}{4}\frac{c\epsilon_0 E_0}{\gamma_0^2} e^{-\frac{1}{2\gamma_0^2}(\omega_0-\omega)^2} \qquad \omega>0}$$