1) Conductivity in Metals

-a]
$$\vec{F}_d = \frac{-m}{\tau}\frac{d\vec{r}}{dt} \qquad [1]$$

using Newton's Law

$$\vec{F}_{ext} + \vec{F}_d = m\frac{d^2\vec{r}}{dt^2} \qquad [2]$$

$$F_{ext} = -e\vec{E}$$

then
$$-e\vec{E} - \frac{m}{\tau}\frac{d\vec{r}}{dt} = m\frac{d^2\vec{r}}{dt^2}$$

$$\frac{d^2\vec{r}}{dt^2} + \frac{1}{\tau}\frac{d\vec{r}}{dt} = \frac{-e}{m}\vec{E}$$

assuming
$$\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$$

then
$$\vec{r} = \vec{r}_0 e^{-i\omega t}$$

$$\left\{(-i\omega)^2 + \frac{(-i\omega)}{\tau}\right\}\vec{r}_0 = \frac{-e}{m}\vec{E}_0$$

$$\vec{r}_0 = \frac{-\frac{e}{m}\vec{E}_0}{-\omega^2 - \frac{i\omega}{\tau}} = \frac{\frac{e}{m}\vec{E}_0}{\omega^2 + \frac{i\omega}{\tau}} \qquad [3]$$

-b] The conductivity [Implies Free charge]

$$\vec{j}(\omega) = \sigma(\omega)\vec{E}(\omega) \qquad [4]$$

$$\vec{j} = -ne\vec{v} \qquad [5]$$

where n is the free electron density.

$$\vec{j} = -ne(-i\omega)\vec{r}_0$$

$$\vec{j} = \frac{+ne[i\omega]\frac{e}{m}\vec{E}_0}{\omega^2 + \frac{i\omega}{\tau}}$$

then
$$\sigma = \frac{i\omega ne^2/m}{\omega^2 + \frac{i\omega}{\tau}} \qquad [6]$$

-c] dielectric constant. The dipole moment is

$$\vec{p} = -e\vec{r}_0$$

then the "polarization" $\vec{P}$ is

$$\vec{P} = n\vec{p} = \frac{-\frac{e^2 n}{m}}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_0$$

then the electric susceptibility is:

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\chi_e = \frac{-e^2 n/m\epsilon_0}{\omega^2 + \frac{i\omega}{\tau}}$$

therefore

$$\boxed{\epsilon(\omega) = 1 + \chi_e = 1 \frac{-\frac{e^2 n}{m\epsilon_0}}{\omega^2 + \frac{i\omega}{\tau}} \qquad [7]}$$

then

$$\boxed{\epsilon(\omega) = 1 + i\frac{\sigma}{\epsilon_0 \omega} \qquad [8]}$$

where σ is defined in eq [13].

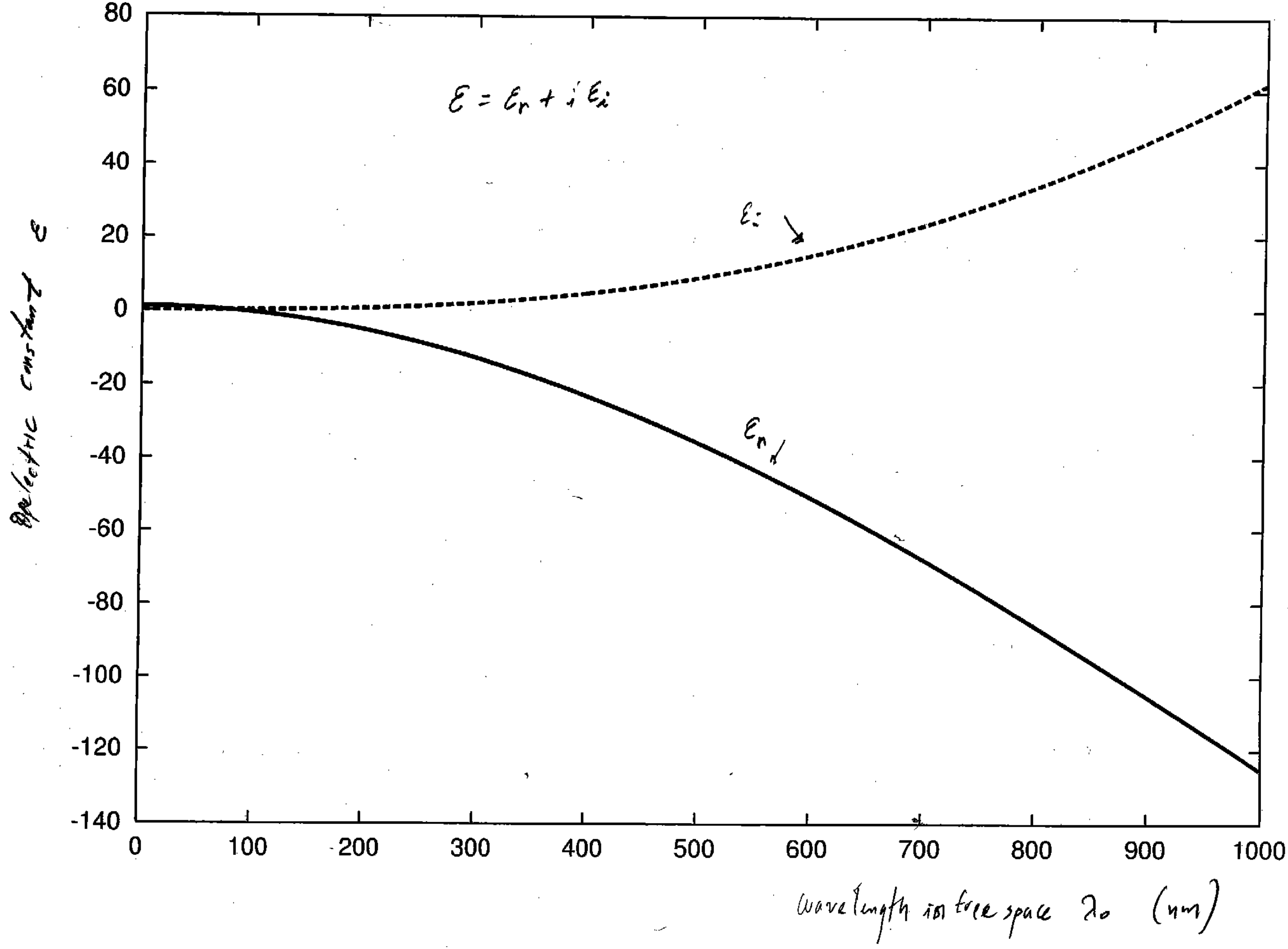
c)

$$\epsilon(\lambda_0) = 1 - \frac{\frac{e^2 n}{m\epsilon_0 c^2}}{\frac{\omega^2}{c^2} + \frac{i\omega}{c^2\tau}}$$

$$\epsilon(\lambda_0) = 1 - \frac{\frac{e^2 n}{m\epsilon_0 c^2}}{\frac{[2\pi]^2}{\lambda_0^2} + i\frac{2\pi}{\lambda_0}\frac{1}{c\tau}}$$

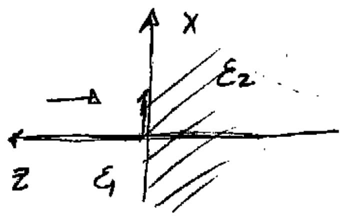
$$\epsilon(\lambda_0) = 1 - \frac{\frac{e^2 n}{m\epsilon_0 c^2}}{\frac{2\pi}{\lambda_0}\left[\frac{2\pi}{\lambda_0} + \frac{i}{c\tau}\right]}$$

See plots



$\varepsilon = \varepsilon_r + i\,\varepsilon_i$
ε_i
ε_r
80
60
40
20
0
-20
-40
-60
-80
-100
-120
-140
0
100
200
300
400
500
600
700
800
900
1000
dielectric constant ε
wavelength in free space λ_0 (nm)

– 1]



$$\vec{E}_i(\vec{r}) = E_0 e^{-i k_0 z}$$

$$E_t(\vec{r}) = t \vec{E}_0 e^{-i k_0 \sqrt{\varepsilon} z} = t \vec{E}_0 e^{-i k_0 (n_R + i n_I) z}$$

$$\vec{E}_t = t \vec{E}_0 e^{-k_0 n_I |z|} e^{-i k_0 n_R z}$$

$\nabla \times E = -i\omega\mu_0 H$

amplitude

$$\vec{E}(z) = t E_0 e^{-k_0 n_I |z|} e^{-i k_0 n_R z} \hat{x}$$

$$\vec{H}(z) = \frac{-i}{\mu_0 \omega} \frac{\partial E_x}{\partial z} \hat{y} = \frac{-i\omega}{\mu_0}\left[-i k_0 n_r + k_0 n_I\right] e^{-k_0 n |z|} e^{-i k_0 n_r z} \hat{y}$$

$$\vec{H}(z) = \frac{t}{\mu\omega}\left[-k_0 n_r - i k_0 n_I\right] e^{-k_0 n_I |z|} e^{-i k_0 n_r z} \hat{z}\, E_0$$

$$\vec{H}^*(z) = \frac{t^*}{\mu_0\omega}\left[-k_0 n_r + i k_0 n_I\right] e^{-k_0 n_I |z|} e^{+i k_0 n_r z} E_0^* \hat{y}$$

$$\langle S\rangle = \frac{1}{2} \mathrm{Re}\{\vec{E} \times \vec{H}^*\} = \frac{\omega k_0}{\mu_0 2} e^{-2 k_0 n_I |z|} |t|^2 \mathrm{Re}\{-n_R\} E_0^2$$

$$|\langle S\rangle| = \frac{1}{2} \frac{k_0}{\mu_0 \omega} |t|^2 |n_R| E_0^2 e^{-2 k_0 n_I |z|}$$

$$|\langle S\rangle| = \frac{1}{2} \varepsilon_0 c\, |t|^2 |n_R|\, E_0^2 e^{-2 k_0 n_I |z|}$$

Therefore

$$I = I_0 e^{-2 k_0 n_I |z|}$$

where $I_0 = \frac{1}{2}\varepsilon_0 c\, |t|^2 |n_R|\, E_0^2$

then $\boxed{\delta = \frac{1}{2 k_0 n_I} = \frac{\lambda_0}{4\pi n_I}}$

δ is the distance in which the intensity drops to I_0/e

For $\lambda_0 = 633$ nm

$\varepsilon = -56.36 + i\, 17.83$

$n = 1.17 + i\, 7.57$

$t = 0.069 - 0.24 i$

$|t|^2 = 0.064$

then $\boxed{\delta = \frac{633}{4\pi(7.59)} \text{ nm} = 6.63 \text{ nm}}$

$\boxed{I_0 = \left(\frac{1}{2} c \varepsilon_0 E_0^2\right)(0.073)}$

For $\lambda_0 = 100$ nm

$\varepsilon = -0.56 + i\, 0.079$

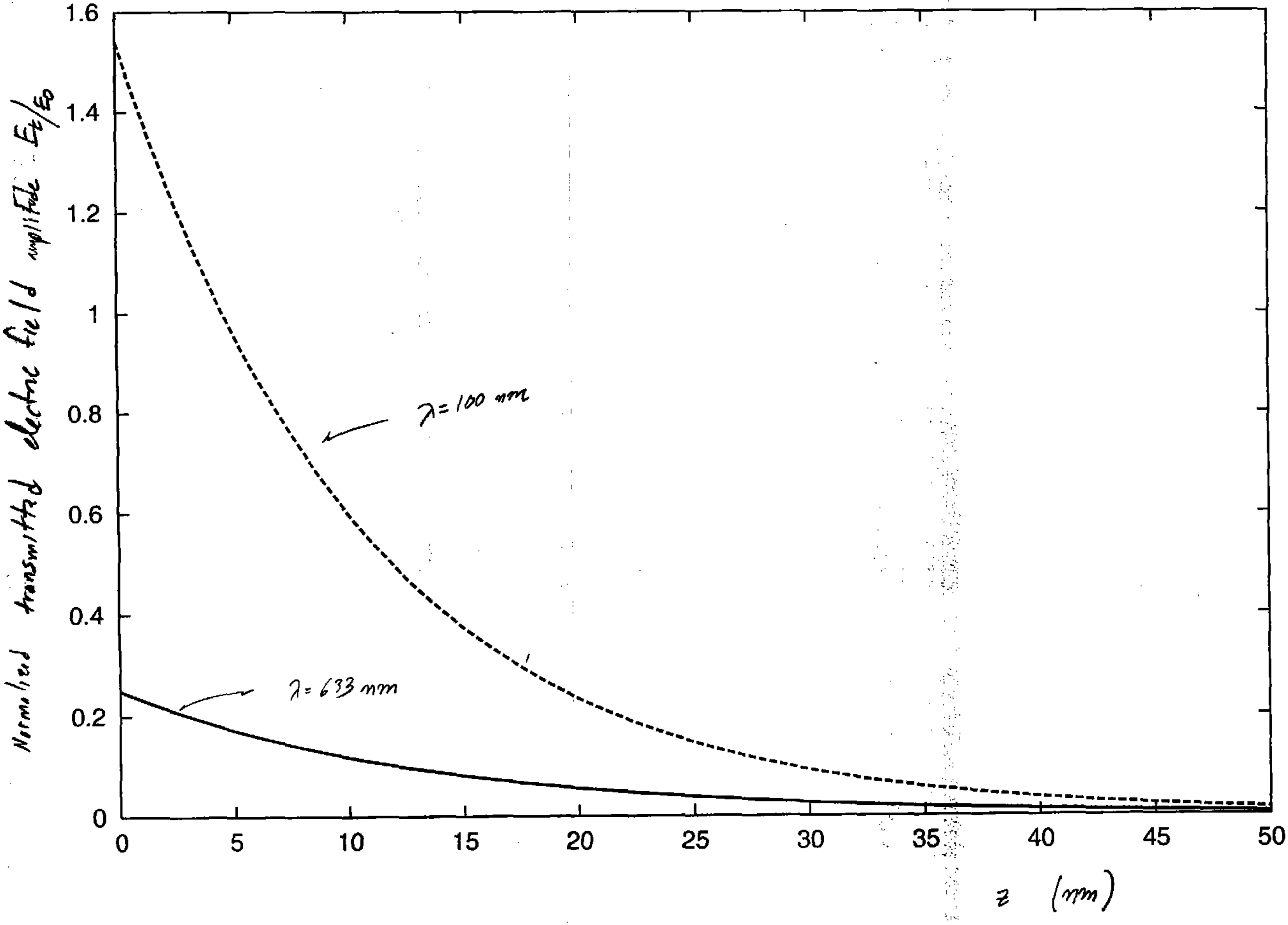
$n = 0.051 + i\, 0.75$

$t = 1.2559 - i\, 0.901$

$|t|^2 = 2.39$

$\delta = \frac{100 \text{ nm}}{4\pi(0.75)} = 10.61$ nm

$I_0 = \left(\frac{1}{2} c \varepsilon_0 E_0^2\right)(0.1218)$



1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35
40
45
50
Normalized transmitted electric field amplitude E_t/E_0
$\lambda = 100$ nm
$\lambda = 633$ nm
z (nm)

2]

a] Diagonalyze the matrix, since it is symmetric all eigenvalues are <u>real</u>

$$\det\begin{bmatrix} 29-\lambda & \sqrt{3} & 6 \\ \sqrt{3} & 31-\lambda & -2\sqrt{3} \\ 6 & -2\sqrt{3} & 20-\lambda \end{bmatrix} = 0$$

solving for the roots of the polynomial

$$\lambda_1 = \lambda_2 = 2 \qquad \lambda_3 = 1$$

Therefore since two axes have the same value, then the crystal is <u>uniaxial</u>

$$\begin{bmatrix} \varepsilon_x = \varepsilon_o & & 0 \\ 0 & \varepsilon_y = \varepsilon_o & 0 \\ 0 & 0 & \varepsilon_z = \varepsilon_e \end{bmatrix}$$

-b] We consider that the x, y, z axes coincide with the principal axes.

It is convenient to work with the $\vec{D}$ vector since $\vec{k}$ and $\vec{D}$ are perpendicular, this fact comes from the divergence condition, namely

$$\nabla \cdot \vec{D} = 0 \qquad [1]$$

if $\vec{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$ [2]

and $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ [3]

then a plane wave has the form

$$\vec{D} = \vec{D}_0 e^{i\vec{k}\cdot\vec{r}}$$

eq. [3] implies

$$\vec{k} \cdot \vec{D}_0 = 0 \qquad [5]$$

Suppose that $\vec{k}$ is the $\hat{z}$ direction i.e. $\vec{k} = k\hat{z}$

From eq [5] results that $\vec{D}_0$ has components on the x-y directions, Furthermore

$$E_{x_0} = \frac{D_{0x}}{\epsilon_0 \varepsilon_x} \qquad E_{y_0} = \frac{D_{0y}}{\epsilon_0 \varepsilon_y}$$

Therefore $\vec{E}$ is also in the x-y plane but since the scale factor for E_{0x} and E_{0y} are different then $\vec{E}_0$ and $\vec{D}$ are <u>not</u> collinear

$\nabla\times\nabla\times E = +i\omega\mu_0\nabla\times H$ (assuming $e^{-i\omega t}$)

$-\nabla^2 E + \cancel{\nabla(\nabla\cdot E)} \overset{0}{=} i\omega\mu_0\nabla\times\vec{H}$

$-\nabla^2 E = i\omega\mu_0(-i\omega)\epsilon_0\vec{D}$

$-\nabla^2 E = \frac{\omega^2}{c^2}\vec{D}$

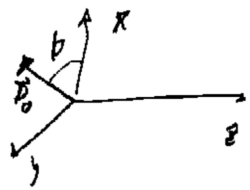
since $\vec{D} = \overset{\leftrightarrow}{\epsilon}E$

then

$-\nabla^2 E = \frac{\omega^2}{c^2}\overset{\leftrightarrow}{\epsilon}\vec{E}$

then the Helmholtz equation is

$$\nabla^2\vec{E} + \frac{\omega^2}{c^2}\overset{\leftrightarrow}{\epsilon}\vec{E} = 0$$



$\nabla^2 E_x + \frac{\omega^2}{c^2}\epsilon_x E_x = 0$

$\nabla^2 E_y + \frac{\omega^2}{c^2}\epsilon_y E_y = 0$

Therefore

$E_x = E_{0x}e^{ik_x z}$

$E_y = E_{0y}e^{ik_y z}$

where $k_x = \frac{\omega}{c}\sqrt{\epsilon_x}$ $\quad k_y = \frac{\omega}{c}\sqrt{\epsilon_y}$

Therefore, if $\sqrt{\epsilon_x} \neq \sqrt{\epsilon_y}$ then the E field with x direction propagates with a different phase velocity than the component of the y direction.

If $\sqrt{\epsilon_x} = \sqrt{\epsilon_y}$ then both components have the same phase velocity