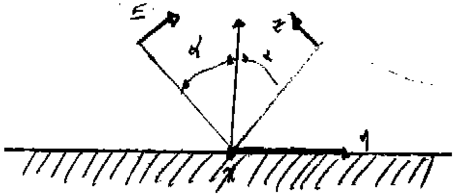
-a]



$$\vec{E}_o(\vec{r}) = \vec{E}_{in}(\vec{r}) + \vec{E}_{rd}(\vec{r})$$

without loss of generality we assumed that $\vec{E}_{in}$ is in the zy plane.

Let

$$\vec{E}_{oi} = E_{oi}\cos\alpha\,\hat{y} + E_{oi}\sin\alpha\,\hat{z}$$

$$\vec{k} = k_o\sin\alpha\,\hat{y} - k_o\cos\alpha\,\hat{z}$$

$$k_o = \frac{2\pi}{\lambda_o} \quad \{\text{free space}\}$$

then

$$\vec{E}_o(\vec{r}) = \vec{E}_o e^{ik_o[\sin\alpha\, y - \cos\alpha\, z]} + \vec{E}_{r_o} e^{ik[\sin\alpha + \cos\alpha\, z]}$$

where

$$\vec{E}_{r_o} = E_{oi} r_p \left[-\cos\alpha\,\hat{y} + \sin\alpha\,\hat{z}\right]$$

$$r_p = \frac{\varepsilon k_{z_1} - k_{z_2}}{\varepsilon k_{z_1} - k_{z_2}}$$

$$k_{z_1} = \sqrt{\frac{\omega^2}{c^2} - k_y^2}$$

$$k_{z_2} = \sqrt{\frac{\omega^2}{c^2}\varepsilon - k_y^2}$$

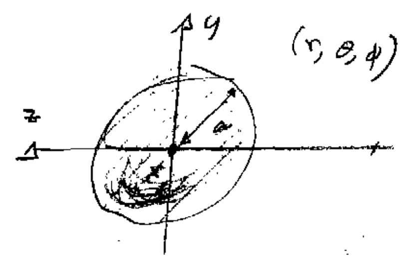
Therefore

$$\vec{E}_o(\vec{r} = z\hat{z}) = \vec{E}_o e^{-ik_o\cos\alpha z} + \vec{E}_{r_o} e^{ik_o\cos\alpha z}$$

$$\boxed{\begin{aligned}\vec{E}_o(\vec{r} = z\hat{z}) = {} & E_{oi}\cos\alpha\left[e^{-ik_o\cos\alpha z} - r_p e^{ik_o\cos\alpha z}\right]\hat{y} \\ & + E_{oi}\sin\alpha\left[e^{-ik_o\cos\alpha z} + r_p e^{ik_o\cos\alpha z}\right]\hat{z}\end{aligned}} \quad [c]$$

-b]

$$\left.\begin{aligned}\nabla\cdot\vec{D} &= 0 \\ \nabla\times\vec{E} &= 0 \\ \nabla\times\vec{H} &= 0 \\ \nabla\cdot\vec{B} &= 0\end{aligned}\right\} \text{STATIC EQUATIONS} \quad [1]$$



From Maxwell's equations we have to solve the Laplace eq:

$$\nabla^2\Phi = 0 \qquad [2]$$

and

$$\vec{E}(\vec{r}) = -\nabla\Phi \qquad [3]$$

<u>Boundary conditions</u> <u>{spherical coordinates}</u>

[i] $\Phi(r\to\infty) = -E_o z = -E_o r\cos\theta$ {uniform field}

$$E_0 \frac{dP_1[\cos\theta]}{d\theta} - \sum_\ell \frac{A_\ell}{a^{\ell+2}} \frac{dP_\ell[\cos\theta]}{d\theta} = -\sum_\ell a^{(\ell-1)} B_\ell \frac{dP_\ell[\cos\theta]}{d\theta}$$

For $\ell = 1$

$$A_1 = B_1 + \frac{E_0}{a^3} \qquad [9]$$

For $\ell \neq 1$

$$A_\ell = B_\ell a^{(2\ell+1)} \qquad [10]$$

From boundary condition [illegible]

$$\vec{D}_\perp = -\varepsilon(r)\epsilon_0 \frac{\partial \Phi}{\partial r}(r=a,\theta)$$ then

$$-\frac{\partial \Phi_2}{\partial r}(r=a,\theta) = -\varepsilon \frac{\partial \Phi_1}{\partial r}$$

$$E_0 P_1 + \sum_\ell \frac{(\ell+1)}{a^{\ell+2}} P_\ell[\cos\theta] = \sum_\ell \left(-\ell a^{\ell-1} \varepsilon B_\ell P_\ell[\cos\theta]\right)$$

for $\ell = 1$

$$E_0 + \frac{2}{a^3} A_1 = -\varepsilon B_1 \qquad [11]$$

for $\ell \neq 1$

$$(\ell+1) A_\ell = -\ell a^{(2\ell-1)} \varepsilon B_\ell \qquad [12]$$

Solving the linear equations [7-12] we find that

$A_\ell = B_\ell = 0$ for $\ell \neq 1$

$$A_1 = \frac{\varepsilon - 1}{\varepsilon + 2} E_0 a^3$$

$$B_1 = \frac{-3}{\varepsilon + 2} E_0$$

Then eq [4] become:

$$\Phi(r,\theta) = \begin{cases} -E_0 r\cos\theta + \frac{\varepsilon-1}{\varepsilon+2} E_0 \frac{a^3}{r^2}\cos\theta & r > a \\ -\frac{3}{\varepsilon+2} E_0 r\cos\theta & r < a \end{cases} \qquad [13]$$

On the other hand the potential for a dipole is given by

$$\Phi_{dip} = \frac{\vec{p}\cdot\vec{r}}{r^3}\frac{1}{4\pi\epsilon_0}$$

If $\vec{p} = p\hat{z}$ at it is located at $\vec{r} = 0$ then

$$\Phi_{dip} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \qquad [14]$$

By comparing with eq [13], The potential outside the sphere is equivalent to a potential produce by an uniform field in the z direction plus the potential produce by a dipole oriented in the same direction as the uniform field whose dipole moment magnitude is

$$p = \frac{4\pi\epsilon_0 a^3 (\varepsilon-1)}{\varepsilon+2} E_0$$

then the polarizability of the particle is given by:

$$\alpha = \frac{p}{E_0} = \frac{4\pi\epsilon_0 a^3 (\varepsilon-1)}{\varepsilon+2}$$

[ii] Tangential components of $\vec{E}$ at $r=a$ are continuos,

$$\vec{E}_{\parallel_1}(r=a) = \vec{E}_{\parallel_2}(r=a)$$

where $\vec{E}_{\parallel_1}$ $\vec{E}_{\parallel_2}$ are the tangential component at the surface in regions outside and inside of the sphere respectively.

[iii] Normal components of $\vec{D}$ are continuos

$$\vec{D}_{\perp_1}(r=a) = \vec{D}_{\perp_2}(r=a)$$

Using spherical coordinates (r,θ,φ). By the geometry Φ does <u>not</u> depend on φ. Therefore it is possible to expand Φ as:

$$\Phi(r,\theta) = \sum_\ell \left\{ \frac{A_\ell}{r^{\ell+1}} + B_\ell r^\ell \right\} P_\ell[\cos\theta]$$

where A_ℓ, B_ℓ are coefficients to be determined and P_ℓ are the Legendre polynomials. Splitting the solution in two regions

$$\Phi(r,\theta) = \begin{cases} \Phi_1(r,\theta) & r<a \\ \Phi_2(r,\theta) & r>a \end{cases} \qquad [4]$$

where

$$\Phi_1(r,\theta) = \sum_\ell B_\ell r^\ell P_\ell[\cos\theta] \qquad [5]$$

$$\Phi_2(r,\theta) = \sum_\ell \left\{ C_\ell r^\ell + \frac{A_\ell}{r^{\ell+1}} \right\} P_\ell[\cos\theta] \qquad [6]$$

Notice that we have dropped the term $r^{-(\ell+1)}$ from eq [5] for $r<a$, since it should not by any discontinuities at $r=0$

From boundary condition [i]

$$\Phi_2(r\to\infty) = -E r\cos\theta$$

$$-E_0 r\cos\theta = \lim_{r\to\infty} \sum_\ell \left\{ C_\ell r^\ell + \cancel{\frac{A_\ell}{r^{\ell+1}}}^{\,0} \right\} P_\ell[\cos\theta]$$

The only term that survives is for $\ell=1$ since

$$P_1[\cos\theta] = \cos\theta$$

therefore

$$C_1 = -E_0 r \qquad [7]$$

$$C_\ell = 0 \quad \text{for } \ell \neq 1 \qquad [8]$$

From boundary condition [ii]

$$\vec{E}_\parallel(\vec{r}=a) = \frac{-1}{a}\frac{\partial \Phi(r=a,\theta)}{\partial\theta}\hat{e}_\theta$$

then

$$-\frac{1}{a}\frac{\partial \Phi_1(r=a,\theta)}{\partial\theta} = -\frac{1}{a}\frac{\partial \Phi_2(r=a,\theta)}{\partial\theta}$$

-c] Following the same procedure as in the Midterm problem 3

$$\vec{p} = \vec{d}_{eff} \cdot \vec{E}_o(\vec{r}_p) \qquad [1] \qquad \vec{r}_p = z_0 \hat{z} \qquad [2]$$

where

$$\vec{d}_{ef} = \alpha \begin{bmatrix} \frac{1}{1-\gamma} & 0 & 0 \\ 0 & \frac{1}{1-\gamma} & 0 \\ 0 & 0 & \frac{1}{1-2\gamma} \end{bmatrix} \qquad [3]$$

where

$$\gamma \equiv \frac{1}{4\pi\epsilon_0} \alpha \frac{\epsilon-1}{(\epsilon+1)} \frac{1}{(2z_0)^3}$$

then if $\alpha = 4\pi\epsilon_0 a^3 \frac{\epsilon-1}{(\epsilon+2)}$

$$\gamma = \frac{1}{8} \frac{(\epsilon-1)^2}{(\epsilon+2)(\epsilon-1)} \left(\frac{a}{z_0}\right)^3 \qquad [4]$$

-d]

Resonances occur when

i] $\gamma = 1$

ii] $\gamma = +1/2$

then for case i

$$1 = \frac{1}{8} \frac{(\epsilon-1)^2}{(\epsilon+2)(\epsilon+1)} \left(\frac{a}{z_0}\right)^3$$

$$\frac{8 z_0^3}{a^3} = \frac{(\epsilon-1)^2}{(\epsilon+2)(\epsilon+1)} = q e^{i\phi}$$

where $q = \left| \frac{(\epsilon-1)^2}{(\epsilon+2)(\epsilon+1)} \right| \qquad \phi = \text{Arg}\left\{ \frac{(\epsilon-1)^2}{(\epsilon+2)(\epsilon+1)} \right\}$

Therefore

$$q = \frac{8 z_0^3}{a^3} \quad \text{and} \quad \phi = 2\pi m \qquad m = 0, \pm 1, \pm 2 \ldots$$

For case ii]

$$q = \frac{4 z_0^3}{a^3} \qquad \phi = 2\pi m \qquad m = 0, \pm 1, \pm 2 \ldots$$

Let

$$\varepsilon = \varepsilon_x + \varepsilon_y i$$

then

$$\frac{(\varepsilon-1)^2}{(\varepsilon+2)(\varepsilon+1)} = \frac{(\varepsilon_x-1)^2 - \varepsilon_y^2 + 2i\,\varepsilon_y(\varepsilon_x-1)}{(\varepsilon_x+2)(\varepsilon_x+1) - \varepsilon_y^2 + i\left[\varepsilon_y(2\varepsilon_x+3)\right]} \qquad [1]$$

$$\mathrm{Arg}\left[\frac{(\varepsilon-1)^2}{(\varepsilon+2)(\varepsilon+1)}\right] = \phi = \mathrm{Arctg}\left[\frac{2\varepsilon_y(\varepsilon_x-1)}{(\varepsilon_x-1)^2-\varepsilon_y^2}\right] - \mathrm{Arctg}\left[\frac{\varepsilon_y(2\varepsilon_x+3)}{(\varepsilon_x+2)(\varepsilon_x-1)-\varepsilon_y^2}\right]$$

$$q = \left|\frac{(\varepsilon-1)^2}{(\varepsilon+2)(\varepsilon+1)}\right| = \frac{\sqrt{\left[[\varepsilon_x-1]^2-\varepsilon_y^2\right]^2 + 4\varepsilon_y^2(\varepsilon_x-1)^2}}{\sqrt{\left[(\varepsilon_x+2)(\varepsilon_x+1)^2-\varepsilon_y^2\right]^2+\varepsilon_y^2(2\varepsilon_x+3)^2}} \qquad [2]$$

See plots for q and ϕ !!

e] Plot for $|\vec{P}|$

Using eq [1] c) and the eq [2] a with

$$e^{i k_0 z \cos\phi} \approx 1$$

then

$$\vec{P} = \alpha \begin{bmatrix} \frac{1}{1-r} & 0 \\ 0 & \frac{1}{1-2\alpha} \end{bmatrix} \begin{bmatrix} E_{0y}(\vec{r} = z_0\hat{z}) \\ E_{0z}(\vec{r} = z_0\hat{z}) \end{bmatrix}$$

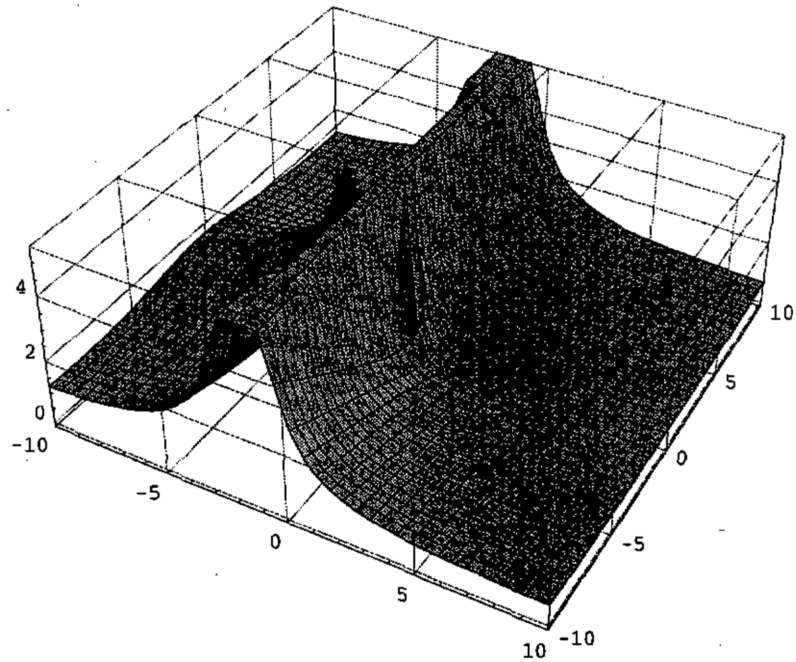
therefore

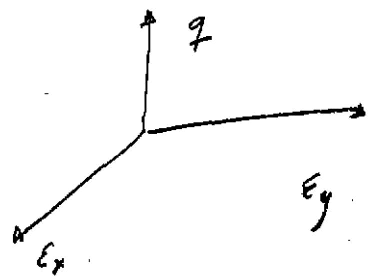
$$|\vec{P}| = \sqrt{(P_x\hat{x} + P_y\hat{y})\cdot(P_x\hat{x}+P_y\hat{y})^*}$$

$$|\vec{P}| = \sqrt{|P_x|^2 + |P_y|^2} \qquad [1]$$

See the plot of eq [1] !

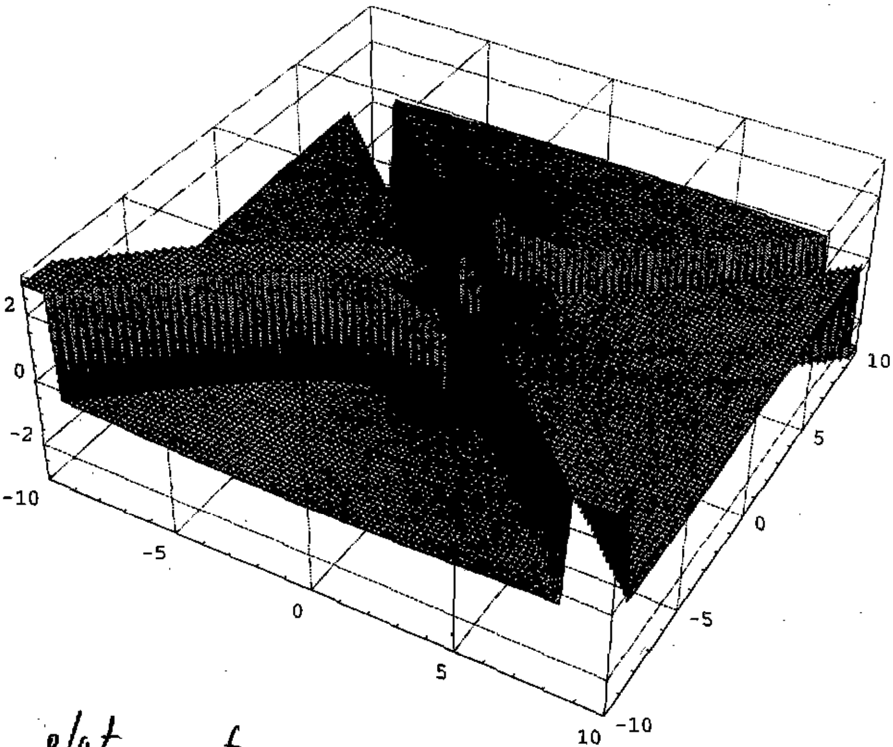
$\mathcal{E} = \mathcal{E}_x + \mathcal{E}_y\, i$

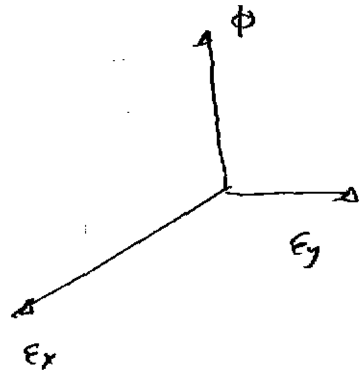




plot of $q(\mathcal{E}_x, \mathcal{E}_y)$

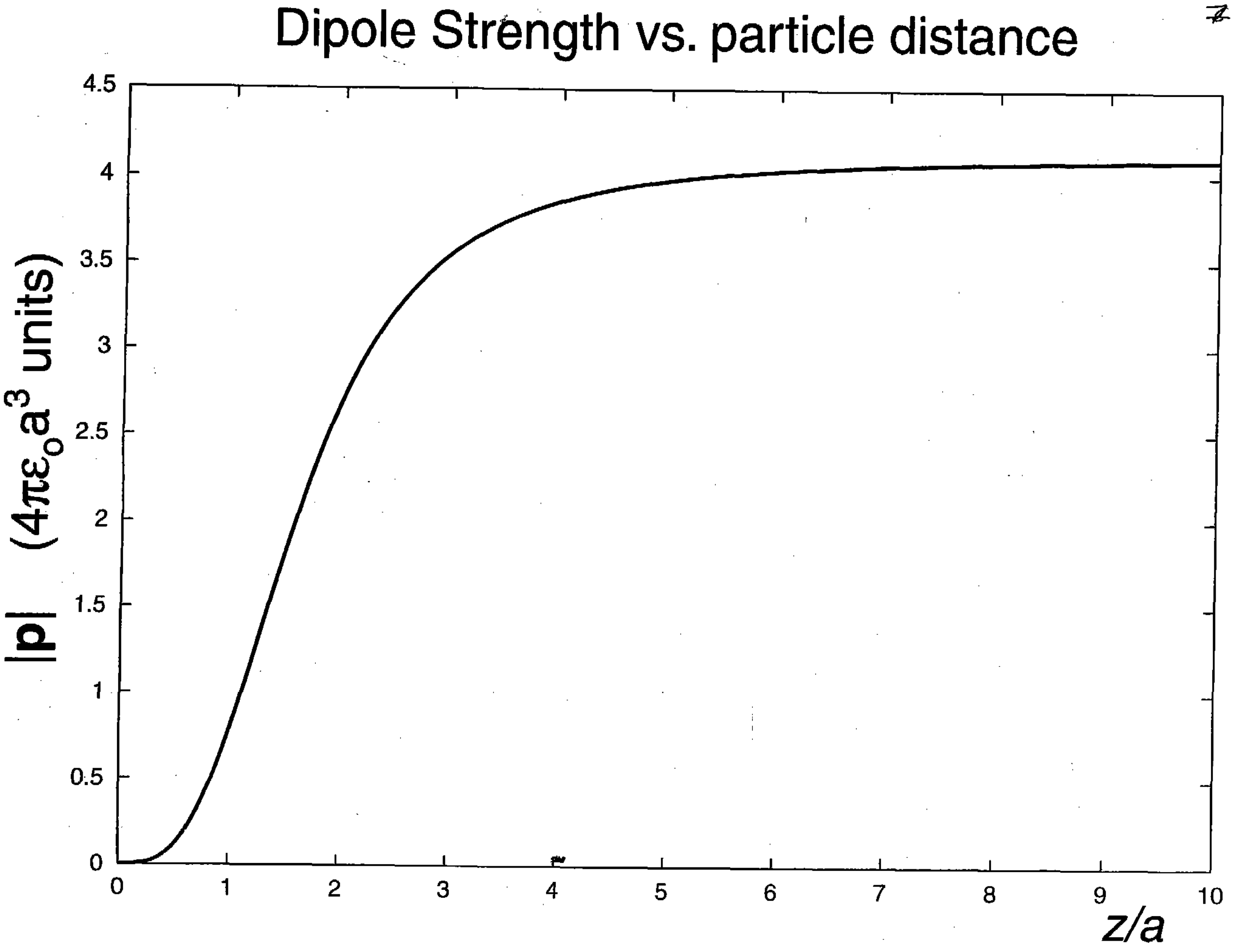
$\varepsilon = \varepsilon_x + i\varepsilon_y$





plot of

$\phi(\varepsilon_x, \varepsilon_y)$



Dipole Strength vs. particle distance
|p| (4πε$_o$a^3 units)
z/a
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
0
1
2
3
4
5
6
7
8
9
10

c] Radiation Patterns

$$\vec{E}_{sc}(r) = \frac{k^2}{\epsilon_0}\sum_{i=1}^{2} \overleftrightarrow{G}(\vec{r},\vec{r}_i)\cdot \vec{p}_i(\vec{r}_i) \qquad [1]$$

$$\vec{E}_{sc}(\vec{r}) = \frac{k^2}{\epsilon_0}\sum_{i=1}^{2} \overleftrightarrow{G}(\vec{r},\vec{r}_i)\cdot \begin{bmatrix} \beta_i\, \vec{E}_{0x}(\vec{r}=z_0\hat{z}) \\ \beta_i\, E_{0y}(\vec{r}=z_0\hat{z}) \\ \eta_i\, E_{0z}(\vec{r}=z_0\hat{z}) \end{bmatrix}$$

where

$$\beta_1 = \frac{\alpha}{1-\gamma} \qquad \beta_2 = -\frac{(\epsilon-1)}{\epsilon+1}\,\alpha\,\frac{1}{1-\gamma}$$

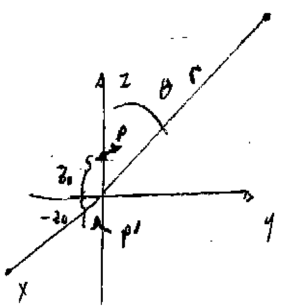
$$\eta_1 = \frac{\alpha}{1-2\gamma} \qquad \eta_2 = \frac{(\epsilon-1)}{(\epsilon+1)}\,\alpha\,\frac{1}{1-2\gamma}$$

Using the Green's function in the far field

$$\overleftrightarrow{G}(\vec{r}, \vec{r}=\pm z_0\hat{z}) = \frac{e^{ikr}}{4\pi r}\, e^{\mp i\frac{k z_0}{r} z} \begin{bmatrix} \frac{x^2}{r^2}-1 & \frac{xy}{r^2} & \frac{xz}{r^2} \\ \frac{xy}{r^2} & \frac{y^2-1}{r^2} & \frac{zy}{r^2} \\ \frac{xz}{r^2} & \frac{yz}{r^2} & \frac{z^2}{r^2}-1 \end{bmatrix}$$

Here we have use the fact that

$r \gg z_0$



Let define

$$\vec{V}_i \equiv \begin{bmatrix} \frac{x^2}{r^2}-1 & \frac{xy}{r^2} & \frac{xz}{r^2} \\ \frac{xy}{r^2} & \frac{y^2-1}{r^2} & \frac{zy}{r^2} \\ \frac{xz}{r^2} & \frac{yz}{r^2} & \frac{z^2}{r^2}-1 \end{bmatrix} \begin{bmatrix} \beta_i E_{0x} \\ \beta_i E_{0y} \\ \eta_i E_{0z} \end{bmatrix}$$

In spherical coordinates

$$V_{ix} = (-1+\sin^2\theta\cos^2\phi)\,\beta_i E_{0x} + \sin^2\theta\cos\phi\sin\phi\,\beta_i E_{0y} + \eta_i \sin\theta\cos\phi\cos\theta\, E_{0z}$$

$$V_{iy} = \sin^2\theta\cos\phi\sin\phi\,\beta_i E_{0x} + (-1+\sin^2\theta\sin^2\phi)\,\beta_i E_{0y} + \eta_i \sin\theta\cos\theta\sin\phi\, E_{0z}$$

$$V_{iz} = \sin\theta\cos\phi\cos\theta\,\beta_i E_{0x} + \sin\theta\cos\theta\sin\phi\,\beta_i E_{0y} - \eta_i \sin^2\theta\, E_{0z}$$

Transforming $\vec{V}$ into spherical components we obtain

$V_{r_i} = 0$ {as expected}

$$V_{\theta_i} = -\beta_i\cos\theta\sin\phi\, E_{0x} - \beta_i\cos\theta\sin\phi\, E_{0y} - \eta_i E_{0z}\sin\theta$$

$$V_{\phi_i} = \beta_i \sin\phi\, E_{0x} - \beta_i\cos\phi\, E_{0y}$$

Then if $E_{0x} = 0$

$$V_{\theta_i} = -\beta_i\cos\theta\sin\phi\, E_{0y} + \eta_i E_{0z}\sin\theta$$

$$V_{\phi_i} = -\beta_i\cos\phi\, E_{0y}$$

From eq [1]

$$\vec{E} = \frac{k^2}{\epsilon_0}\frac{1}{4\pi}\frac{e^{ikr}}{r}\Big\{ E_{0y}\cos\theta\sin\phi\left[\beta_1 e^{-ik\cos\theta z_0} + \beta_2 e^{ik\cos\theta z_0}\right]\hat{e}_\theta$$

$$+ E_{0z}\sin\theta\left[\eta_1 e^{-ik\cos\theta z_0} + \eta_2 e^{ik\cos\theta z_0}\right]\hat{e}_\theta$$

$$- E_{0y}\cos\phi\left[\beta_1 e^{-ik\cos\theta z_0} + \beta_2 e^{-ik\cos\theta z_0}\right]\hat{e}_\phi$$

$$\langle\vec{S}\rangle = \frac{1}{2}\,\mathrm{Re}\{\vec{E}\times\vec{H}^*\}$$

$$\vec{H} = -\frac{i c^2\epsilon_0}{\omega}\nabla\times\vec{E} = -\frac{i c^2\epsilon_0}{\omega}\left\{-ik E_\phi\,\hat{e}_\theta + ik E_\theta\,\hat{e}_\phi\right\}$$

$$\vec{H} = \frac{c^2\epsilon_0 k}{\omega}\left\{-E_\phi\hat{e}_\theta + E_\theta\hat{e}_\phi\right\}$$

$$\vec{H}^* = \frac{c^2\epsilon_0 k}{\omega}\left\{-E_\phi^*\hat{e}_\theta + E_\theta^*\hat{e}_\phi\right\}$$

$$\langle\vec{S}\rangle = \frac{1}{2}\frac{c^2\epsilon_0 k}{\omega}\,\hat{e}_r\left[|E_\theta|^2 + |E_\phi|^2\right]$$

therefore

$$\frac{dP}{d\Omega} = \frac{1}{2}\frac{c^2\epsilon_0 k}{\omega}r^2\left[|E_\theta|^2 + |E_\phi|^2\right]$$

assuming $kz_0 \ll 1$

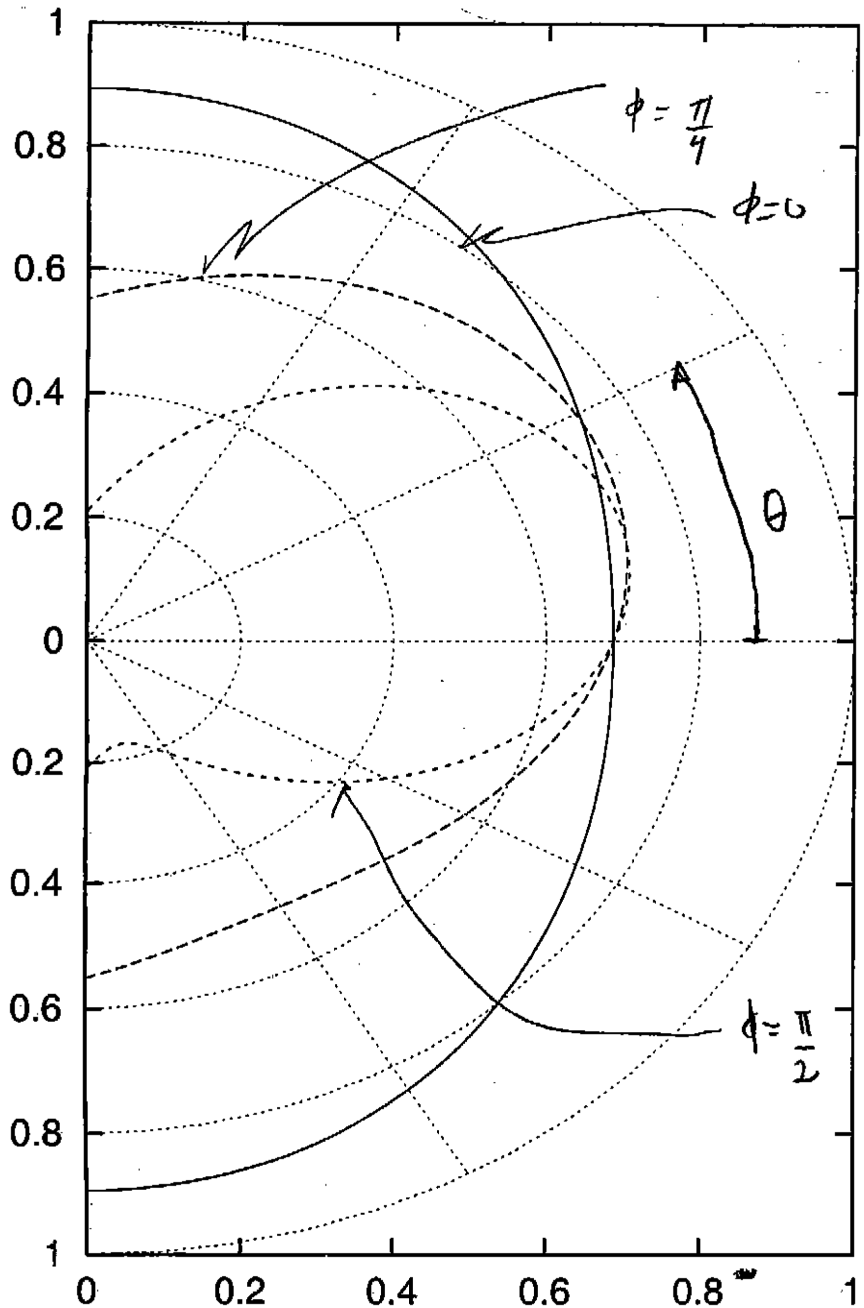
$$\frac{dP}{d\Omega} = \frac{k_0^4 c}{(4\pi)^2\epsilon_0}\Big[|E_{0y}|^2\cos^2\theta\sin^2\phi\left(|\beta_1|^2+|\beta_2|^2+2\mathrm{Re}\{\beta_1\beta_2^*\}\right)$$

$$+|E_{0z}|^2\sin^2\theta\left[|\eta_1|^2+|\eta_2|^2+2\mathrm{Re}\{\eta_1\eta_2^*\}\right]$$

$$2\cos\theta\sin\phi\sin\theta\,\mathrm{Re}\{E_{0y}(\beta_1+\beta_2)E_{0z}^*(\eta_1^*+\eta_2^*)\}$$

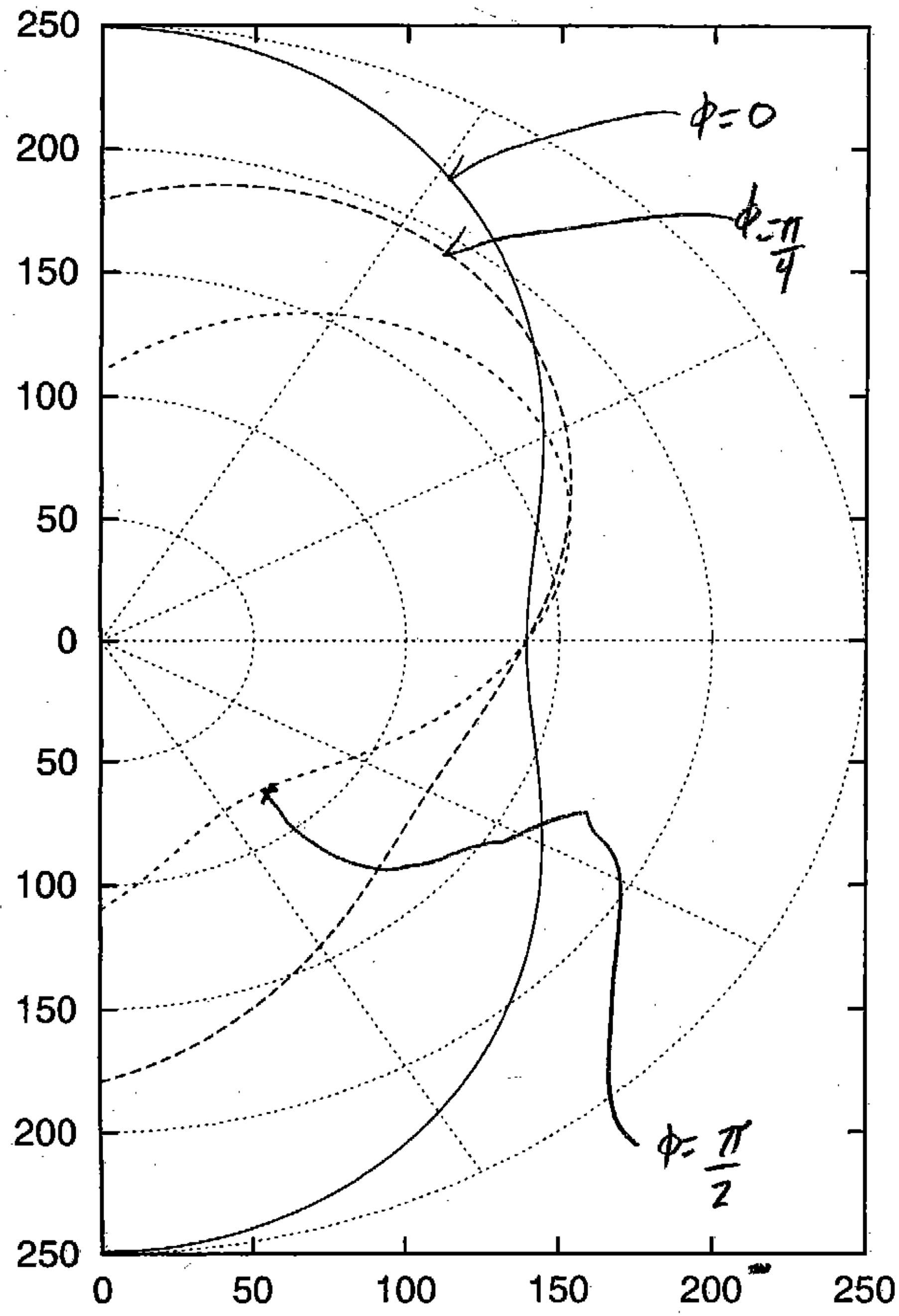
$$+|E_{0y}|^2\cos^2\phi\left(|\beta_1|^2+|\beta_2|^2+2\mathrm{Re}\{\beta_1\beta_2\}\right)\Big]$$

see plots



Polar plot of the power emited by solid angle in $\frac{k_0 c E_0^2 \alpha'^2}{4(4\pi)^2 \epsilon_0}$ units $(\alpha' = 4\pi\epsilon_0 a^3)$

$\frac{dP(\theta,\phi)}{d\Omega}$ at $\frac{z}{a} = 0,5$

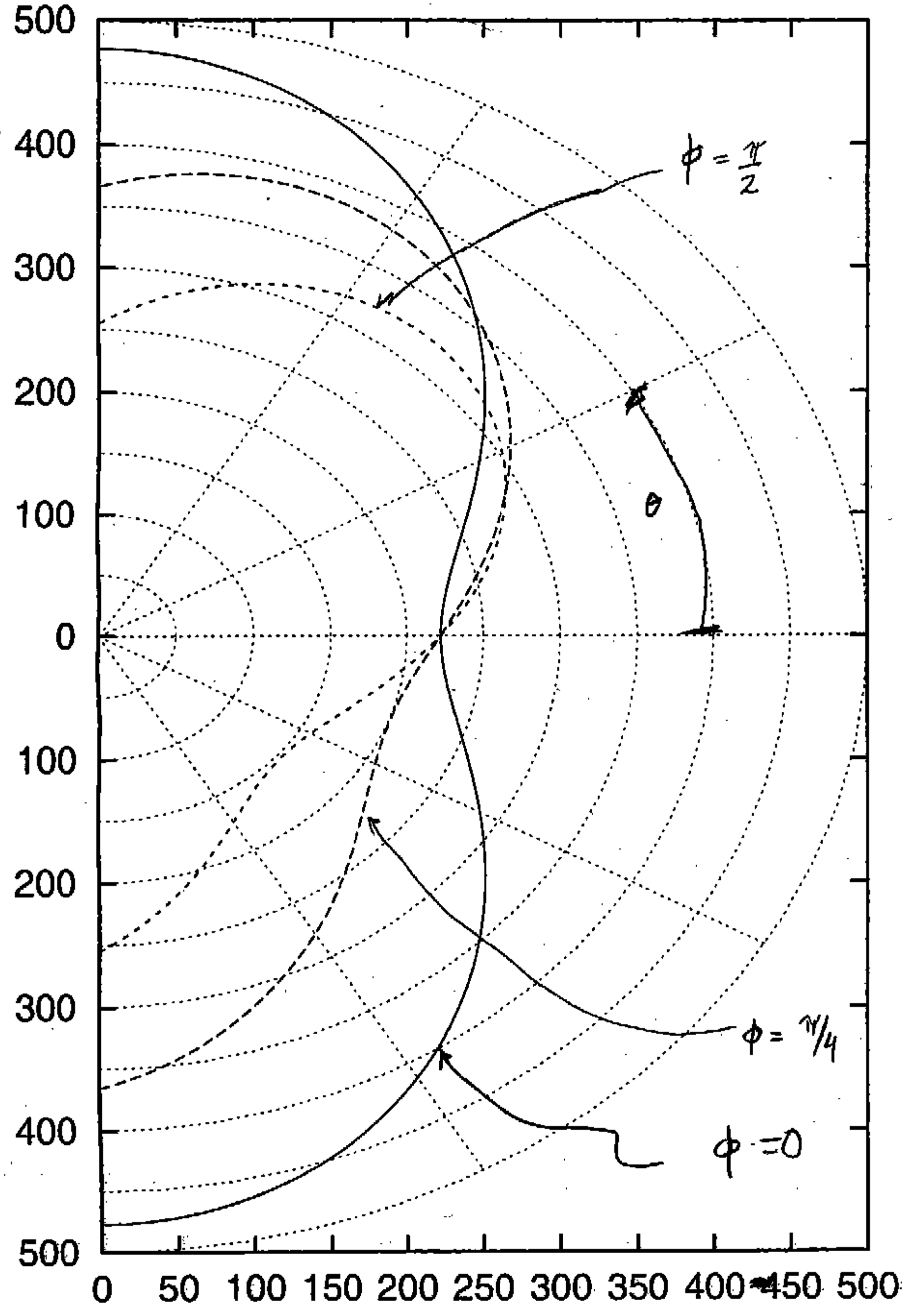


Polar plot of the power emitted by solid angle in

$$\frac{k_0 c E_0^2 r^2}{4(4\pi)^2 \epsilon_0}$$ units ($\alpha' = 4\pi$ to a^3)

$$\frac{dP(\theta, \phi)}{d\Omega} \quad \text{at} \quad \frac{z}{a} = 2$$

Polar plot of the power emitted by solid angle in $\frac{k_0^4 d E_0^2 q^2}{4(4\pi)^2\epsilon_0}$ units $\alpha' = 4\pi\epsilon_0 a^3$



$$\frac{dP(\theta, \phi)}{d\Omega}$$

at $\frac{z}{a} = 10$ $k z_0 \ll 1$