1) If $\vec{A}$ and ϕ are in the Lorentz Gauge, then

$$\nabla\cdot\vec{A}_L = -\frac{1}{c^2}\frac{\partial\phi_L}{\partial t} \qquad [1]$$

Now we define the new Coulomb gauge as:

$$\vec{A}_c = \vec{A}_L + \nabla g \quad [2] \qquad \phi_c = \phi_L - \frac{\partial g}{\partial t} \quad [3]$$

Where g is a scalar function and $\nabla\cdot A_c = 0$

therefore

$$\nabla\cdot A_c = \nabla\cdot\vec{A}_L + \nabla^2 g = 0 \qquad [4]$$

using eqs [4], and [1] - eq [3] becomes

$$-\nabla^2 g = -\frac{1}{c^2}\frac{\partial\phi_c}{\partial t} - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2}$$

$$\nabla^2 g + \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = +\frac{1}{c^2}\frac{\partial\phi_c}{\partial t} \qquad [5]$$

Furthermore since $\nabla^2\phi_c = -\rho/\epsilon_0$

the solution of ϕ_c is given by:

$$\phi_c = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}',t)\, d^3\vec{r}'}{|\vec{r}-\vec{r}'|} \qquad [6]$$

Therefore

$$\nabla^2 g + \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = f(\vec{r},t) \qquad [7]$$

where $f(\vec{r},t)$ is defined as:

$$f(\vec{r},t) = \frac{1}{4\pi\epsilon_0 c^2}\int\frac{\frac{\partial\rho(\vec{r}',t)}{\partial t}}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}' \qquad [8]$$

$$g(\vec{r},t) = \frac{1}{(4\pi)^2\epsilon_0 c^2}\int\frac{[f(\vec{r}',t)]_{retarded}\, d^3\vec{r}'}{|\vec{r}-\vec{r}'|} \qquad [9]$$

<u>other possible solution</u>

Using eq [4] into eq [1]

$$\nabla^2 g = -\frac{1}{c^2}\frac{\partial\phi_L}{\partial t} \qquad [10]$$

Furthermore,

$$g = \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int\frac{\phi_L(\vec{r}',t)\, d^3 r'}{|\vec{r}-r'|} \qquad [11]$$

Furthermore

$$\nabla^2\phi_L - \frac{1}{c^2}\ddot{\phi}_L = \frac{-\rho}{\epsilon_0}$$

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{dr'^3}{|\vec{r}-\vec{r}'|}[\rho[\vec{r}',t']]_{retarded}$$

$$g = \frac{1}{16\pi^2 c^2\epsilon_0}\frac{\partial}{\partial t}\iint\frac{[\rho(\vec{r}'',t')]_{retarded}}{|\vec{r}-\vec{r}'|\,|\vec{r}'-\vec{r}''|}\, d^3\vec{r}'\, d^3\vec{r}'' \qquad [12]$$

2.1) Show that

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + r'^2 - 2(\vec{r}\cdot\vec{r}')} \approx r - \frac{\vec{r}\cdot\vec{r}'}{r} \qquad [1]$$

Let

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \qquad [2]$$

$$\vec{r}' = x'\hat{x} + y'\hat{y} + z'\hat{z} \qquad [3]$$

$$|\vec{r}-\vec{r}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$|\vec{r}-\vec{r}'| = \sqrt{x^2+y^2+z^2 + x'^2+y'^2+z'^2 - 2xx' - 2yy' - 2zz'}$$

$$|\vec{r}-\vec{r}'| = \sqrt{r^2 + r'^2 - 2\,\vec{r}\cdot\vec{r}'}$$

where $r^2 = |\vec{r}|^2$ and $r'^2 = |\vec{r}'|^2$

$$|\vec{r}-\vec{r}'| = r\sqrt{1 + \frac{r'^2}{r^2} - \frac{2\vec{r}\cdot\vec{r}'}{r^2}}$$

using Taylor expansion

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

$$|\vec{r}-\vec{r}'| \approx r\left[1 + \frac{r'^2}{2r^2} - \frac{\vec{r}\cdot\vec{r}'}{r^2}\right]$$

since $\frac{r'^2}{2r^2} \ll \frac{\vec{r}\cdot\vec{r}'}{r^2}$

then

$$\boxed{|\vec{r}-\vec{r}'| \approx r - \frac{\vec{r}\cdot\vec{r}'}{r}}$$

2.2 LINE CURRENT

Charge Density

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \qquad [1]$$

If $\vec{J}$ and ρ have time harmonic dependence $e^{-i\omega t}$ then

$$\nabla\cdot\vec{J}(\vec{r}) = i\omega\rho(\vec{r})$$

so

$$\rho(\omega) = -\frac{i}{\omega}\nabla\cdot\vec{J}(\vec{r}) \qquad [2]$$

Since

$$\vec{J}(\vec{r}') = I_0\cos\left[\frac{(2n+1)}{2z_0}\pi z'\right]\delta(x')\delta(y')\,\hat{n}_z \qquad |z|\le z_0$$

From eq [2]

$$\rho(\vec{r}') = \frac{-i}{\omega}\frac{\partial}{\partial z'}\left\{I_0\cos\left[\frac{(2n+1)\pi z'}{2z_0}\right]\delta(x')\delta(y')\right\} \qquad |z|\le z_0$$

$$\boxed{\rho(\vec{r}') = \frac{i}{\omega}\frac{(2n+1)\pi}{2z_0}\sin\left[\frac{(2n+1)\pi z'}{2z_0}\right]\delta(x')\delta(y') \qquad |z|\le z_0} \qquad [3]$$

Scalar potential

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho(\vec{r}')e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}d^3r' \qquad [4]$$

Since

$$\vec{r} = x\hat{x}+y\hat{y}+z\hat{z} \qquad \vec{r}' = z'\hat{z} \qquad d^3r' = dx'dy'dz'$$

Substituting eq [3] into eq [4], and solving the integral with respect to x', y',

$$\phi(\vec{r}) = \frac{iI_0}{4\pi\epsilon_0\omega}\cdot\frac{(2n+1)\pi}{2z_0}\int_{-z_0}^{z_0}\sin\left[\frac{(2n+1)\pi z'}{2z_0}\right]\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dz' \qquad [5]$$

Approximating the Green's function according to problem [2.1]

$$\phi(\vec{r}) = \frac{iI_0}{4\pi\epsilon_0\omega}\frac{\beta}{r}e^{ikr}\int_{-z_0}^{z_0}e^{-i\alpha z'}\sin[\beta z']\,dz' \qquad [6]$$

we have defined

$$\beta\equiv\frac{(2n+1)\pi}{2z_0} \quad\text{and}\quad \alpha\equiv\frac{kz}{r} \qquad [7]$$

then

$$\phi(\vec{r}) = \frac{iI_0}{4\pi\epsilon_0\omega}\frac{\beta}{r}e^{ikr}\int_{-z_0}^{z_0}\frac{e^{-i\alpha z'}}{2i}\left\{e^{i\beta z'}-e^{-i\beta z'}\right\}dz'$$

$$\phi(\vec{r}) = \frac{iI_0}{4\pi\epsilon_0\omega}\frac{\beta}{r}e^{ikr}\cdot\frac{1}{2i}\left\{\frac{e^{i(\beta-\alpha)z'}}{i(\beta-\alpha)}-\frac{e^{-i(\beta+\alpha)z'}}{-i(\beta+\alpha)}\right\}\Bigg|_{z'=-z_0}^{z'=z_0}$$

$$\phi(\vec{r}) = \frac{iI_0}{4\pi\epsilon_0\omega}\frac{\beta}{r}e^{ikr}\frac{1}{2i}\left\{\frac{e^{i(\beta-\alpha)z_0}-e^{-i(\beta-\alpha)z_0}}{i(\beta-\alpha)}+\frac{e^{-i(\beta+\alpha)z_0}-e^{i(\beta+\alpha)z_0}}{i(\beta+\alpha)}\right\}$$

$$\phi(\vec r) = \frac{i I_0}{4\pi\epsilon_0\omega}\frac{\beta}{r\, i}e^{ikr}\left\{\frac{\sin[(\beta-\alpha)z_0]}{\beta-\alpha} - \frac{\sin[(\beta+\alpha)z_0]}{(\beta+\alpha)}\right\}$$

$$\phi(r) = \frac{I_0}{4\pi\epsilon_0\omega}\frac{\beta}{r}\frac{e^{ikr}}{\beta^2-\alpha^2}\left\{\beta\left[\sin[(\beta-\alpha)z_0] - \sin[(\beta+\alpha)z_0]\right]\right.$$

$$\left. + \alpha\left\{\sin[(\beta-\alpha)z_0 + \sin[(\beta+\alpha)z_0]\right\}\right\}$$

$$\phi(\vec r) = \frac{I_0}{4\pi\epsilon_0\omega}\frac{\beta}{r}\frac{e^{ikr}}{\beta^2-\alpha^2}\left\{-2\beta\cos[\beta z_0]\sin[\alpha z_0] + 2\alpha\sin[\beta z_0]\cos[\alpha z_0]\right\}$$

since

$$\left.\begin{aligned}\cos[\beta z_0] &= \cos\left[\frac{(2n+1)}{2}\pi\right] = 0\\ \sin[\beta z_0] &= \sin\left[\frac{(2n+1)\pi}{2}\right] = (-1)^n\end{aligned}\right\} \quad [8]$$

then

$$\phi(\vec r) = \frac{I_0}{2\pi\epsilon_0\omega}\frac{\beta}{r}\frac{e^{ikr}}{\beta^2-\alpha^2}(-1)^n\alpha\cos[\alpha z_0]$$

using spherical coordinates (r,θ,ϕ)

$$\alpha = \frac{kz}{r} = k\cos\theta$$

therefore

$$\phi(\vec r) = \frac{(-1)^n I_0\beta\, k\cos\theta\; e^{ikr}\cos[k\cos\theta\, z_0]}{2\pi\epsilon_0\omega\,(\beta^2-k^2\cos^2\theta)\,r} \quad [9]$$

Vector Potential

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}\int J(\vec r')\frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|}d^3\vec r' \quad [10]$$

$$\vec J(\vec r') = I_0\cos\left[\frac{(2n+1)\pi z'}{2z_0}\right]\delta(x')\delta(y')\,\hat n_{z'} \qquad |z|\le z_0 \quad [11]$$

Here

$\vec r = x\hat x + y\hat y + z\hat z \qquad \vec r' = z'\hat z \qquad d\vec r^3 = dx'dy'dz'$

Substituting eq [11] into eq [10] and solving for x' and y'

$$\vec A(\vec r) = \frac{\mu_0}{4\pi}I_0\int_{-z_0}^{z}\cos\left[\frac{(2n+1)}{2z_0}\pi z'\right]\frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|}dz'\,\hat n_z$$

using the result of problem 2.1

$$A(\vec r) = \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\int_{-z_0}^{z_0}\cos\left[\frac{(2n+1)}{2z_0}\pi z'\right]e^{-i\frac{kz}{r}z'}dz'\,\hat n_z$$

by the definition of eq [7]

$$\vec A(\vec r) = \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\int_{-z_0}^{z_0}\cos[\beta z']e^{-i\alpha z'}dz'\,\hat n_z$$

$$\vec A(\vec r) = \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\int_{-z_0}^{z_0}\frac{e^{-i\alpha z'}}{2}\left\{e^{i\beta z'} + e^{-i\beta z'}\right\}dz'\,\hat n_z$$

$$\vec A(\vec r) = \frac{\mu_0 I_0}{4\pi}\frac{e^{ikr}}{r}\int_{-z_0}^{z_0}\frac{1}{2}\left(e^{i(\beta-\alpha)z'} + e^{-i(\beta+\alpha)z'}\right)dz'\,\hat n_z$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0}{4\pi r} \frac{e^{ikr}}{2} \left\{ \frac{e^{i(\beta-\alpha)z'}}{i(\beta-\alpha)} - \frac{e^{-i(\beta+\alpha)z'}}{i(\beta+\alpha)} \right\} \Bigg|_{z=-z_0}^{z=z_0} \hat{n}_z$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0}{4\pi r} \frac{e^{ikr}}{2i} \left[\frac{e^{i(\beta-\alpha)z_0} - e^{-i(\beta-\alpha)z_0}}{(\beta-\alpha)} - \frac{e^{-i(\beta+\alpha)z_0} - e^{i(\beta+\alpha)z_0}}{(\beta+\alpha)} \right] \hat{n}_z$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0}{4\pi r} e^{ikr} \left\{ \frac{\sin[(\beta-\alpha)z_0]}{\beta-\alpha} + \frac{\sin[(\beta+\alpha)z_0]}{\beta+\alpha} \right\} \hat{n}_z$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0}{4\pi r} \frac{e^{ikr}}{\beta^2-\alpha^2} \Big[\beta\{\sin[(\beta-\alpha)z_0] + \sin[(\beta+\alpha)z_0]\}$$

$$+\alpha\{\sin[(\beta-\alpha)z_0] - \sin[(\beta+\alpha)z_0]\} \Big] \hat{n}_z$$

$$A(\vec{r}) = \frac{\mu_0 I_0}{4\pi r} \frac{e^{ikr}}{\beta^2-\alpha^2} \left\{ 2\beta \sin[\beta z_0]\cos[\alpha z_0] - 2\alpha \cos[\beta z_0]\sin[\alpha z_0] \right\} \hat{n}_z$$

since $\sin[\beta z_0] = (-1)^n$ and $\cos[\beta z_0] = 0$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I (-1)^n \beta e^{ikr}}{2\pi r (\beta^2-\alpha^2)} \cos[\alpha z_0] \hat{n}_z \qquad [12]$$

using spherical coordinates (r, θ, ϕ) eq [12] becomes

$$\vec{A}(\vec{r}) = \frac{(-1)^n \mu_0 I_0}{2\pi} \frac{\beta e^{ikr}}{r} \frac{\cos[k\cos\theta z_0]}{\beta^2 - k^2\cos^2\theta} \hat{n}_z$$

since $\hat{n}_z = \cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta$

$$\vec{A}(\vec{r}) = \frac{(-1)^n \mu_0 I_0}{2\pi} \frac{\beta e^{ikr}}{r} \frac{\cos[k\cos\theta z_0]}{\beta^2 - k^2\cos^2\theta} \left\{ \cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta \right\} \qquad [13]$$

Magnetic "far" field

$$\vec{H}_{1/r} = \frac{1}{\mu_0}(\nabla \times \vec{A})_{1/r} = \frac{1}{\mu_0} \hat{e}_\phi \frac{1}{r} \frac{\partial}{\partial r}[r A_\theta] \qquad [14]$$

therefore

$$\vec{H}_{1/r} = \frac{(-1)^n (-i) I_0}{2\pi} \frac{\beta k e^{ikr}}{r} \frac{\cos[k\cos\theta z_0]\sin\theta}{\beta^2 - k^2\cos^2\theta} \hat{e}_\phi \qquad [15]$$

Electric "far" field

$$\vec{E}(\vec{r})_{1/r} = \mu_0 c\, \vec{H}_{1/r} \times \hat{e}_r$$

$$\vec{E}(\vec{r})_{1/r} = \frac{\mu_0 c (-1)^n (-i) I_0}{2\pi} \frac{\beta k e^{ikr}}{r} \frac{\cos[k\cos\theta z_0]\sin\theta}{\beta^2 - k^2\cos^2\theta} \hat{e}_\theta \qquad [16]$$

Power emitted per stereo-radian

$$p(\theta,\phi) = \langle \vec{S} \rangle \cdot r^2 \hat{e}_r = \frac{1}{2} Re\{ \vec{E} \times \vec{H}^* \} r^2 \hat{e}_r \qquad [17]$$

From eqs [15] and [16]

$$p(\theta,\phi) = \frac{\mu_0 c}{8\pi^2} \frac{I_0^2 \beta^2 k^2 \cos^2[k\cos\theta z_0] \sin^2[\theta]}{(\beta^2 - k^2\cos^2\theta)^2}$$

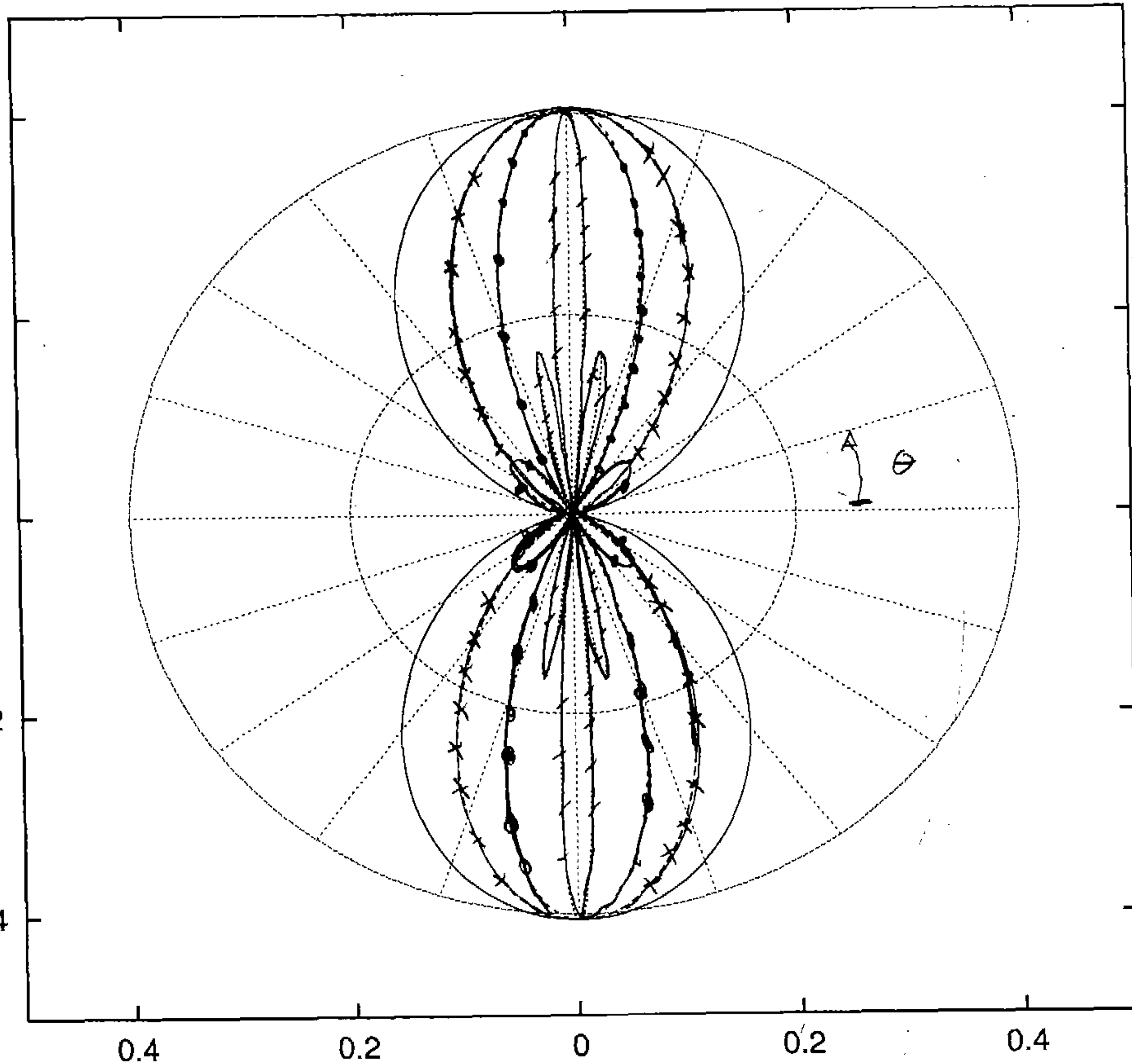
$$p(\theta,\phi) = \frac{I_0^2 \beta^2 k^2}{8\pi^2 c \epsilon_0} \frac{\cos^2[k\cos\theta z_0] \sin^2\theta}{(\beta^2 - k^2\cos^2\theta)^2} \qquad [18]$$

Total emitted power

$$P_T = \int p(\theta,\phi)\, d\Omega = \int_0^{\pi}\int_0^{2\pi} p(\theta,\phi) \sin\theta \, d\phi \, d\theta$$

$$P_T = \frac{I_0 \beta^2 k^2}{4\pi \epsilon_0 c} \int_0^{\pi} \frac{\cos^2[k\cos\theta z_0] \sin^3\theta \, d\theta}{(\beta^2 - k^2\cos^2\theta)^2} \qquad [19]$$

Power emitted per sr $P(\theta)$ for $n=0$ in a polar plot



— $kz_0 = 0.01$

—×— $kz_0 = \frac{1}{2}\pi$

—•— $kz_0 = \pi$

—/— $kz_0 = 5\pi$

$$\frac{P(\theta)}{\frac{I_0^2 \gamma^2 q^2}{8\pi\epsilon_0 c}}$$ } normalized power

where

$q = kz_0$

$\gamma = \beta z_0$

2.3 RING CURRENT

Charge Density

$$\nabla\cdot\bar{J} = -\frac{\partial\tilde{\rho}}{\partial t} \quad [1]$$

If $\vec{J}$ and $\tilde{\rho}$ are time harmonic $e^{-i\omega t}$

$$\nabla\cdot\vec{J}(\vec{r}) = i\omega\,\tilde{\rho}(\vec{r})$$

$$\tilde{\rho}(\vec{r}) = \frac{-i}{\omega}\,\nabla\cdot\vec{J}(\vec{r}) \quad [2]$$

since

$$\vec{J}(\vec{r}') = I_0 \cos[n\phi']\,\delta(\rho'-\rho_0)\,\delta(z)\,\hat{n}_{\phi'} \quad [3]$$

the From eq [2]

$$\tilde{\rho}(\vec{r}') = \frac{-i}{\rho'\omega} I_0 \frac{\partial}{\partial\phi'}\left\{\cos[n\phi']\,\delta(\rho'-\rho_0)\,\delta(z)\right\}$$

$$\boxed{\tilde{\rho}(\vec{r}) = \frac{i I_0 n}{\rho'\omega}\sin[n\phi']\,\delta(\rho'-\rho_0)\,\delta(z)} \quad [4]$$

Scalar Potential

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'} \tilde{\rho}(\vec{r}')\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,d^3r' \quad [5]$$

$$\left.\begin{aligned} d^3r &= \rho\,d\phi\,d\theta\,dz \qquad \vec{r} = \rho\hat{e}_\rho + z\hat{z} \\ \vec{r}' &= \rho_0'\,\hat{e}_\rho' \end{aligned}\right\} \quad [6]$$

using eq [4] and eq [6], and integrating with respect to ρ' and z'

$$\phi(\vec{r}) = \frac{i I_0 n}{4\pi\epsilon_0\omega}\int_0^{2\pi}\sin[n\phi]\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\,d\phi'$$

using the approximation of problem 2.1

and by the fact that

$$\vec{r}\cdot\vec{r}' = \rho_0\rho\cos[\phi'-\phi] \quad [6\,b]$$

$$\phi(\vec{r}) = \frac{i I_0 n}{4\pi\epsilon_0\omega}\frac{e^{ikr}}{r}\int_0^{2\pi} e^{-ik\rho\rho_0\cos[\phi'-\phi]}\sin[n\phi]\,d\phi' \quad [7]$$

using the following identities.

$$\left.\begin{aligned} \int_0^{2\pi}\cos m\gamma\, e^{ix\cos[\gamma-\delta]}\,d\gamma &= 2\pi i^m J_m(x)\cos(m\delta) \\ \int_0^{2\pi}\sin m\gamma\, e^{ix\cos[\gamma-\delta]}\,d\gamma &= 2\pi i^m J_m(x)\sin(m\delta) \end{aligned}\right\} \quad [8]$$

J_m are the Bessel functions of order m, Thus eq [7] is

$$\phi(\vec{r}) = \frac{i I_0 n}{2\epsilon_0\omega}\frac{e^{ikr}}{r}\,i^n\sin[n\phi]\,J_n\left[\frac{-k\rho_0\rho}{r}\right] \quad [9]$$

using the fact that

$$J_n(-x) = (-1)^n J_n(x) \quad [10]$$

$$\psi(\vec{r}) = \frac{i^{n+1} I_0\, n\, (-1)^n}{2\epsilon_0\, r} e^{ikr} \sin[n\phi]\, J_n\left[\frac{k\rho}{r}\rho_0\right]$$

using spherical coordinates $\frac{\rho}{r} \equiv \sin\theta$

then
$$\phi(\vec{r}) = \frac{i^{n+1}(-1)^n I_0\, n}{2\epsilon_0\, r} e^{ikr} \sin[n\phi]\, J_n\left[k\sin\theta\, \rho_0\right] \qquad [11]$$

Vector Potential

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\, e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\, d^3\vec{r}' \qquad [12]$$

$$\vec{J}(\vec{r}') = I_0 \cos[n\phi']\,\delta(\rho'-\rho_0)\,\delta(z')\,\{-\sin\phi'\hat{x} + \hat{y}\cos\phi'\} \qquad [13]$$

Substituting eq [13] into [12] and using the definitions of eq [6] and integrating trivially with respect to ρ' z'

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0 \rho_0}{4\pi}\int_0^{2\pi} \cos[n\phi']\,\{-\sin\phi'\,\hat{x} + \cos\phi'\,\hat{y}\}\, \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\, d\phi'$$

Using the approximation of problem 2.1 and eq [6b]

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0 \rho_0}{4\pi}\frac{e^{ikr}}{r}\int_0^{2\pi} \cos[n\phi']\{-\sin\phi'\hat{x} + \cos\phi'\hat{y}\}\, e^{-i\frac{k\rho\rho_0}{r}\cos[\phi'-\phi]}\, d\phi'$$

Since
$$\cos[n\phi']\cos\phi' = \frac{1}{2}\{\cos[(n+1)\phi'] + \cos[(n-1)\phi']\} \qquad [14]$$

$$\cos[n\phi']\sin\phi' = \frac{1}{2}\{\sin[(n+1)\phi'] - \sin[(n-1)\phi']\} \qquad [15]$$

So

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0 \rho_0}{8\pi}\frac{e^{ikr}}{r}\int_0^{2\pi} e^{-i\frac{k\rho\rho_0}{r}\cos[\phi'-\phi]}\Big(\{-\sin[(n+1)\phi'] + \sin[(n-1)\phi']\}\hat{x}$$
$$+ \{\cos[(n+1)\phi'] + \cos[(n-1)\phi']\}\hat{y}\Big)\, d\phi'$$

using the identities of eq [8]

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0 \rho_0}{4}\frac{e^{ikr}}{r}\Big(\{i^{n+1}\sin[(n+1)\phi]\,J_{n+1}\left[-\frac{k\rho_0\rho}{r}\right] + (i)^{n+1}\sin[(n-1)\phi']\,J_{n-1}\left[-\frac{k\rho_0\rho}{r}\right]\}\hat{x}$$
$$+ \{i^{n+1}\cos[(n+1)\phi]\,J_{n+1}\left[-\frac{k\rho\rho}{r}\right] + i^{n-1}\cos[(n-1)\phi]\,J_{n-1}\left[-\frac{k\rho}{r}\rho_0\right]\}\hat{y}\Big)$$

Using eq [10] and using the fact that

$$\frac{\rho}{r} = \sin\theta$$

$$A(\vec{r}) = \frac{\mu_0 I_0 \rho_0}{4}\frac{e^{ikr}}{r} i^n \Big(\{i(-1)^{n+1}\sin[(n+1)\phi]\,J_{n+1}[k\rho_0\sin\theta] - i(-1)^{n-1}\sin[(n-1)\phi]\,J_{n-1}[k\sin\theta\rho_0]\}\hat{x}$$
$$+ \{(-1)^n i\cos[(n+1)\phi]\,J_{n+1}[k\rho_0\sin\theta] - (-1)^{n-1} i\cos[(n-1)\phi]\,J_{n-1}[k\rho_0\sin\theta]\}\hat{y}\Big)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I_0 \rho_0}{4r} e^{ikr} i^{n+1} (-1)^{n+1} \left[f(\theta,\phi)\hat{x} + g(\theta,\phi)\hat{y} \right] \quad [16]$$

where

$$f[\theta,\phi] = -\left[\sin[(n+1)\phi] J_{n+1}[k\rho_0 \sin\theta] + \sin[(n-1)\phi] J_{n-1}[k\rho_0 \sin\theta] \right]$$

$$f[\theta,\phi] = -\sin n\phi \cos\phi \left[J_{n+1}[k\rho_0 \sin\theta] + J_{n-1}[k\rho_0 \sin\theta] \right]$$

$$- \cos n\phi \sin\phi \left[J_{n+1}[k\rho_0 \sin\theta] - J_{n-1}[k\rho_0 \sin\theta] \right]$$

using the following recurrence relations

$$\left. \begin{aligned} J_{n+1}[x] + J_{n-1}[x] &= \frac{2n}{x} J_n[x] \\ -J_{n+1}[x] + J_{n-1}[x] &= 2 \frac{d}{dx} J_n(x) \end{aligned} \right\} \quad [17]$$

$$f[\theta,\phi] \equiv \frac{-2n}{k\rho_0 \sin\theta} \sin n\phi \cos\phi \, J_n[k\rho_0 \sin\theta] + 2 \cos n\phi \sin\phi \frac{d \, J_n[k\rho_0 \sin\theta]}{d(k\rho_0 \sin\theta)} \quad [18]$$

and

$$g[\theta,\phi] = \cos[(n+1)\phi] J_{n+1}[k\rho_0 \sin\theta] - \cos[(n-1)\phi] J_{n-1}[k\rho_0 \sin\theta]$$

$$g[\theta,\phi] = \cos n\phi \cos\phi \left[J_{n+1}[k\rho_0 \sin\theta] - J_{n-1}[k\rho_0 \sin\theta] \right]$$

$$- \sin n\phi \cos\phi \left[J_{n+1}[k\rho_0 \sin\theta] + J_{n-1}[k\rho_0 \sin\theta] \right]$$

Applying the same recurrence relation of eq [17]

$$g[\theta,\phi] \equiv -\frac{2n \sin n\phi \sin\phi}{k\rho_0 \sin\theta} J_n[k\rho_0 \sin\theta] - 2 \frac{d \, J_n[k\rho_0 \sin\theta]}{d(k\rho_0 \sin\theta)} \cos n\phi \cos\phi \quad [19]$$

Transforming eq [16] into spherical coordinates

$$A(\vec{r}) = \mu_0 \frac{I_0 \rho_0}{4r} e^{ikr} (-i)^{n+1} \Big\{ \sin\theta \left[\cos\phi \, f[\theta,\phi] + \sin\phi \, g[\theta,\phi] \right] \hat{e}_r$$

$$+ \cos\theta \left[\cos\phi \, f[\theta,\phi] + \sin\phi \, g[\theta,\phi] \right] \hat{e}_\theta$$

$$+ \left(\cos\phi \, g[\theta,\phi] - \sin\phi \, f[\theta,\phi] \right) \hat{e}_\phi$$

By trivial algebra

$$\vec{A}(\vec{r}) = (-i)^{n+1} \frac{\mu_0 I_0 \rho_0}{4r} e^{ikr} \Bigg[\frac{n \sin(n\phi)}{k\rho_0} J_n[k\rho_0 \sin\theta] \hat{e}_r$$

$$+ \frac{n \sin(n\phi) \cos\theta}{k\rho_0 \sin\theta} J_n[k\rho_0 \sin\theta] \hat{e}_\theta$$

$$+ \cos n\phi \frac{d \, J_n[k\rho_0 \sin\theta]}{d(k\rho_0 \sin\theta)} \hat{e}_\phi \Bigg] \quad [20]$$

Magnetic Far Field

$$\vec{H}_{1/r} = \frac{1}{\mu_0} (\nabla \times \vec{A})_{\frac{1}{r}} = \frac{1}{\mu_0}\left[\hat{e}_\theta \left\{ -\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\right\} + \hat{e}_\phi \left[\frac{1}{r}\frac{\partial}{\partial r}(r A_\theta)\right]\right]$$

then

$$\vec{H}_{1/r} = \frac{I_0 \rho_0 (-i)^{n+1} i k}{2r}\left\{ -\cos(n\phi)\frac{d\, J_n[k\rho_0 \sin\theta]}{d(k\rho_0\sin\theta)}\hat{e}_\theta + \frac{n \sin(n\phi)\cos\theta\, J_n[k\rho_0 \sin\theta]}{k\rho_0\sin\theta}\hat{e}_\phi\right.$$

[21]

Electric Far Field

$$E(\vec{r})_{1/r} = \mu_0 c\, \vec{H}_{1/r} \times \hat{e}_r$$

$$\vec{E}(\vec{r})_{1/r} = \frac{c\mu_0 I_0 (-i)^{n+1} i k \rho_0}{2r}\left\{\frac{n\sin(n\phi)\cos\theta\, J_n[k\rho_0\sin\theta]}{k\rho_0\sin\theta}\hat{e}_\theta - \cos n\phi \frac{d\, J_n[k\rho_0\sin\theta]}{d(k\rho_0\sin\theta)}\hat{e}_\phi\right\}$$

[22]

Power emitted per Sr

$$p(\theta,\phi) = \frac{1}{2}\,\mathrm{Re}\,\{E \times H^*\} \cdot \hat{e}_r\, r^2 \qquad [23]$$

Then substituting eqs [22] and [21] into eq [23]

$$p(\theta,\phi) = \frac{1}{8}\frac{I_0^2 \rho_0^2 k^2}{c\epsilon_0}\left\{\cos^2 n\phi \left[\frac{d\, J_n(k\rho_0\sin\theta)}{d(k\rho_0\sin\theta)}\right]^2 + \frac{n^2\sin^2(n\phi)}{(k\rho_0\sin\theta)^2} J_n^2[k\rho_0\sin\theta]\right\} \qquad [24]$$

Total emitted Power

$$P_T = \int_0^{\pi}\int_0^{2\pi} p(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

therefore from eq [24]

$$P_T = \frac{\pi}{8}\frac{I_0^2\rho_0^2 k^2}{c\epsilon_0}\int_0^{\pi}\left\{\left[\frac{d\, J_n[k\rho_0\sin\theta]}{d(k\rho_0\sin\theta)}\right]^2 + \frac{n^2}{k^2\rho_0^2\sin^2\theta} J_n^2[k\rho_0\sin\theta]\right\}\sin\theta\, d\theta$$

[25]