1) Inhomogeneous medium in (1-D) z direction

Wave equation for $\vec{E}$ field assuming $e^{-i\omega t}$ harmonic time dependence, then Maxwells equations are for linear medium and non-magnetic [no free charges and currents]

$$\nabla \times \vec{E}(\vec{r}) = i\omega \vec{B}(\vec{r}) \quad [1]$$
$$\nabla \times \vec{B} = -i\mu_0 \epsilon_0 \omega \epsilon(\vec{r}) \vec{E}(\vec{r}) \quad [2]$$
$$\nabla \cdot [\epsilon(\vec{r})\vec{E}] = 0 \quad [3]$$
$$\nabla \cdot \vec{B} = 0 \quad [4]$$

$$-\nabla^2 E(\vec{r}) + \nabla(\nabla \cdot \vec{E}) = i\omega \nabla \times B(\vec{r})$$

$$-\nabla^2 E(\vec{r}) + \nabla(\nabla \cdot \vec{E}) = i\omega \left[-i\mu \epsilon_0 \epsilon(\vec{r}) \vec{E}(\vec{r})\right]$$

$$-\nabla^2 E(\vec{r}) + \nabla(\nabla \cdot \vec{E}) = \frac{\omega^2}{c^2} \epsilon(\vec{r}) \vec{E}(\vec{r})$$

From eq [3]

$$\nabla \cdot [\epsilon(\vec{r})\vec{E}(\vec{r})] = \vec{E}(\vec{r}) \cdot \nabla \epsilon(\vec{r}) + \epsilon(\vec{r}) \nabla \cdot \vec{E}(\vec{r}) = 0$$

then

$$\nabla \cdot E(\vec{r}) = -\vec{E}(\vec{r}) \cdot \frac{\nabla \epsilon(\vec{r})}{\epsilon(\vec{r})} = -\vec{E}(\vec{r}) \cdot \nabla[\ln \epsilon(\vec{r})]$$

therefor the wave equation for $\vec{E}$

$$\boxed{\nabla^2 E(\vec{r}) + \nabla\left[E(\vec{r}) \cdot \nabla[\ln \epsilon(\vec{r})]\right] + \frac{\omega^2}{c^2} \epsilon(\vec{r}) \vec{E}(\vec{r}) = 0} \quad [5]$$

The magnetic wave equation is:

$$-\nabla^2 \vec{B} = -i\mu_0 \epsilon_0 \omega \nabla \times [\epsilon(\vec{r}) \vec{E}(\vec{r})]$$

$$-\nabla^2 \vec{B} = \frac{-i}{c^2} \omega \left\{ \nabla \epsilon(\vec{r}) \times E(\vec{r}) + \epsilon(\vec{r}) \nabla \times E \right\}$$

$$-\nabla^2 B = \frac{-i}{c^2} \omega \left[\frac{\nabla \epsilon(\vec{r}) \times [\nabla \times B(\vec{r})]}{-i\mu_0 \epsilon_0 \omega \epsilon(\vec{r})} \right] + \epsilon(\vec{r}) i\omega \vec{B}(\vec{r})$$

$$-\nabla^2 \vec{B} = \nabla[\ln \epsilon(\vec{r})] \times [\nabla \times B(\vec{r})] + \frac{\omega^2}{c^2} \epsilon(\vec{r}) \vec{B}(\vec{r})$$

$$\boxed{\nabla^2 \vec{B} + \nabla[\ln \epsilon(\vec{r})] \times [\nabla \times \vec{B}(\vec{r})] + \frac{\omega^2}{c^2} \epsilon(\vec{r}) \vec{B}(\vec{r}) = 0} \quad [6]$$

assuming inhomogeneity only in z direction we propose the following solution

TE modes:

$$\vec{E}(\vec{r}) = E(z)\, e^{i(k_x x + k_y y)} \hat{e}_{xy} \quad [7]$$

TM modes

$$\vec{B}(\vec{r}) = B(z)\, e^{i(k_x x + k_y y)} \hat{e}_{xy} \quad [8]$$

Here $\hat{e}_{xy}$ is an arbitrary unit vector in xy plane.

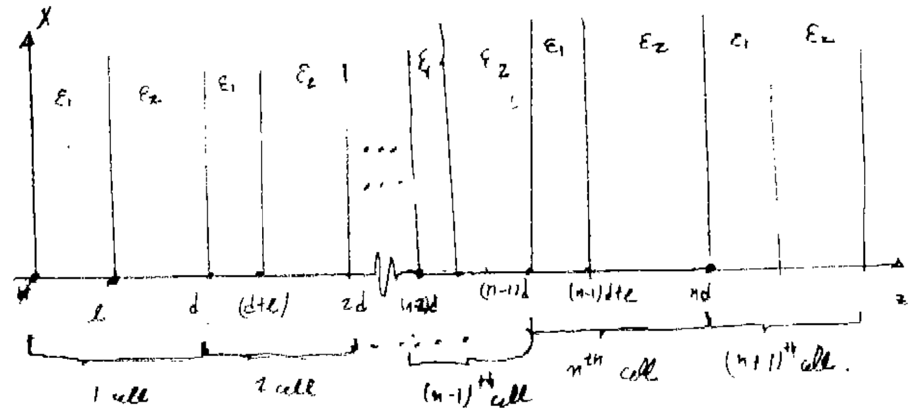
Substituting eqs [7] and [8] into their respective wave equation we obtain

For TE

$$\frac{d^2E(z)}{dz^2} + \left(\frac{\omega^2}{c^2}\epsilon(z) - k_x^2 - k_y^2\right)E(z) = 0 \qquad [9]$$

For TM

$$\frac{d^2B(z)}{dz^2} - \frac{1}{\epsilon(z)}\frac{d\epsilon(z)}{dz}\frac{dB(z)}{dz} + \left[\frac{\omega^2}{c^2}\epsilon(z) - k_x^2 - k_y^2\right]B(z) = 0 \qquad [10]$$



Since there is a homogeneous medium in each of the stack than the wave equations [9] and [10] became:

For TE

$$\frac{d^2E^{(n)j}(z)}{dz^2} + \left[\frac{\omega^2}{c^2}\epsilon_j - k_x^2 - k_y^2\right]E^{nj}(z) = 0 \qquad [11]$$

For TM

$$\frac{d^2B^{nj}(z)}{dz^2} + \left[\frac{\omega^2}{c^2}\epsilon_j - k_x^2 - k_y^2\right]B^{nj}(z) = 0 \qquad [12]$$

where n is the cell number and $j = 1, 2$ refers to medium ϵ_1 or ϵ_2.

Their respective solutions of eqs [11] and [12] can be expressed as:

- TE

$$E^{nj}(z) = A^{nj} e^{ik_j(x-nd)} + B^{nj} e^{-ik_j(x-nd)} \qquad [13]$$

- TM

$$B^{nj}(z) = A^{nj} e^{ik_j(x-nd)} + B^{nj} e^{-ik_j(x-nd)} \qquad [14]$$

Here we have defined

$$k_j \equiv \sqrt{\frac{\omega^2}{c^2}\epsilon_j - k_\parallel^2} \qquad [15]$$

$$k_\parallel^2 \equiv k_x^2 + k_y^2 \qquad [16]$$

We try to related the coefficients of the same material of the n^{th} and $(n+1)^{th}$ cell. Now we apply boundary conditions, the tangential components of the $\vec{E}$ are continuous in any of the boundary, this is true for the $\vec{B}$ field since the medium is non magnetic.

For TE modes

$$E^{(n-1)2}[z=(n-1)d] = E^{(n)1}[z=(n-1)d]$$

$$\frac{dE^{(n-1)2}}{dz}[z=(n-1)d] = \frac{dE^{n1}}{dz}[z=(n-1)d]$$

* The tangential components of B are proportional to $\frac{dE}{dz}$

For TM modes

$$B^{(n-1)2}\left[z=(n-1)d\right] = B^{n1}\left[z=(n-1)d\right]$$

$$\frac{1}{\epsilon_2}\frac{dB^{(n-1)2}[z=(n-1)d]}{dz} = \frac{1}{\epsilon_1}\frac{dB^{n1}[z=(n-1)d]}{dz}$$

* Tangential compents of $\bar{B}$ are proportional to $\frac{1}{\epsilon_j}\frac{dB^{nj}}{dz}$

From the definitions of the E(z) and B(z) fields [eq. [13] and [14]]

$$A^{(n-1)2} + B^{(n-1)2} = A^{n1}e^{-ik_1d} + Be^{ik_1d} \qquad [17a]$$

$$A^{(n-1)2} - B^{(n-1)2} = p_m\left\{A^{n1}e^{-ik_1d} - Be^{ik_1d}\right\} \qquad [17b]$$

where p_m $(m=1,2)$ is a factor that depends on the polarization

$p_1 \equiv K_1/k_2$ TE modes

$p_2 \equiv \frac{\epsilon_2}{\epsilon_1}\frac{k_1}{k_2}$ TM modes

The Boundaries conditions but now at $z=(n-1)d+l$ becomes

$$A^{n1}e^{-ik_1(d-l)} + B^{n1}e^{ik_1(d-l)} = A^{n2}e^{-ik_2(d-l)} + B^{n2}e^{ik_2(d-l)} \qquad [18a]$$

$$A^{n1}e^{-ik_1(d-l)} - B^{n1}e^{ik_1(d-l)} = \frac{1}{p_m}\left\{A^{n2}e^{-ik_2(d-l)} - B^{n2}e^{ik_2(d-l)}\right\} \qquad [18b]$$

Writing eqs [17a] and [17b] and eqs [18a] and [18b] in using matrices then

$$\mathbb{A}\,x^{(n-1)2} = \mathbb{B}\,x^{n1} \qquad [19a]$$

$$\mathbb{C}\,x^{n1} = \mathbb{D}\,x^{n2} \qquad [19b]$$

where

$$\mathbb{A} \equiv \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbb{B} = \begin{bmatrix} e^{-ik_1d} & e^{ik_1d} \\ p_m e^{-ik_1d} & -p_m e^{ik_1d} \end{bmatrix}$$

$$\mathbb{C} \equiv \begin{bmatrix} e^{-ik_1(d-l)} & e^{ik_1(d-l)} \\ e^{-ik_1(d-l)} & -e^{ik_1(d-l)} \end{bmatrix} \qquad \mathbb{D} = \begin{bmatrix} e^{-ik_2(d-l)} & e^{ik_2(d-l)} \\ \frac{1}{p_m}e^{-ik_2(d-l)} & \frac{1}{p_m}e^{ik_2(d-l)} \end{bmatrix}$$

and

$$x^{nj} \equiv \begin{bmatrix} A^{nj} \\ B^{nj} \end{bmatrix}$$

We know can express x^{n2} in terms of $x^{(n-1)2}$, therefore so ....

$$\boxed{x^{(n-1)2} = \mathbb{F}\,x^{n2}} \qquad [20]$$

where $\boxed{\mathbb{F} \equiv \mathbb{A}^{-1}\mathbb{B}\,\mathbb{C}^{-1}\mathbb{D}}$ [21]

by some algebra $\mathbb{F}$ reduces to

$$F_{11} = e^{-i k_2 (d-\ell)} \left\{ \cos(k_1 \ell) - \frac{1}{2} i \left[\frac{1}{\rho_m} + \rho_m\right] \sin(k_1 \ell) \right\} \quad [22a]$$

$$F_{12} = \frac{i}{2} e^{i k_2 (d-\ell)} \left\{ \rho_m - \frac{1}{\rho_m} \right\} \sin k_1 \ell \quad [22b]$$

$$F_{21} = F_{12}^{*} \quad [22c]$$

$$F_{22} = F_{11}^{*} \quad [22d]$$

Now by the Bloch theorem,

$$\mathbb{X}^{(n-1)2} = e^{-i k_{B2} d} \mathbb{X}^{n2} \quad [23]$$

therefore combining eqs [20] and [23]

$$\mathbb{F} \mathbb{X}^{n2} = e^{-i k_B d} \mathbb{X}^{n2}$$

$$\left[\mathbb{F} - e^{-i k_B d} \mathbb{I}\right] \mathbb{X}^{n2} = 0 \quad [24]$$

where $\mathbb{I}$ is the identity matrix

we obtain the dispersion relations by looking for non-trivial solutions of eq [24], so,

$$\det\left[\mathbb{F} - e^{-i k_B d} \mathbb{I}\right] = 0 \quad [25]$$

we obtain that the result of eq [25] is:

$$\cos k_{B2} d = \cos[k_1 \ell] \cos[k_2 (d-\ell)] - \frac{1}{2}\left[\rho_m + \frac{1}{\rho_m}\right] \sin[k_1 \ell] \sin[k_2 (d-\ell)]$$

Therefore for TE

$$\rho_1 = \frac{k_1}{k_2}$$

so TE

$$\cos k_{B2} d = \cos[k_1 \ell] \cos[k_2 (d-\ell)] - \frac{1}{2}\left[\frac{k_1}{k_2} + \frac{k_2}{k_1}\right] \sin[k_1 \ell] \sin[k_2 (d-\ell)] \quad [26]$$

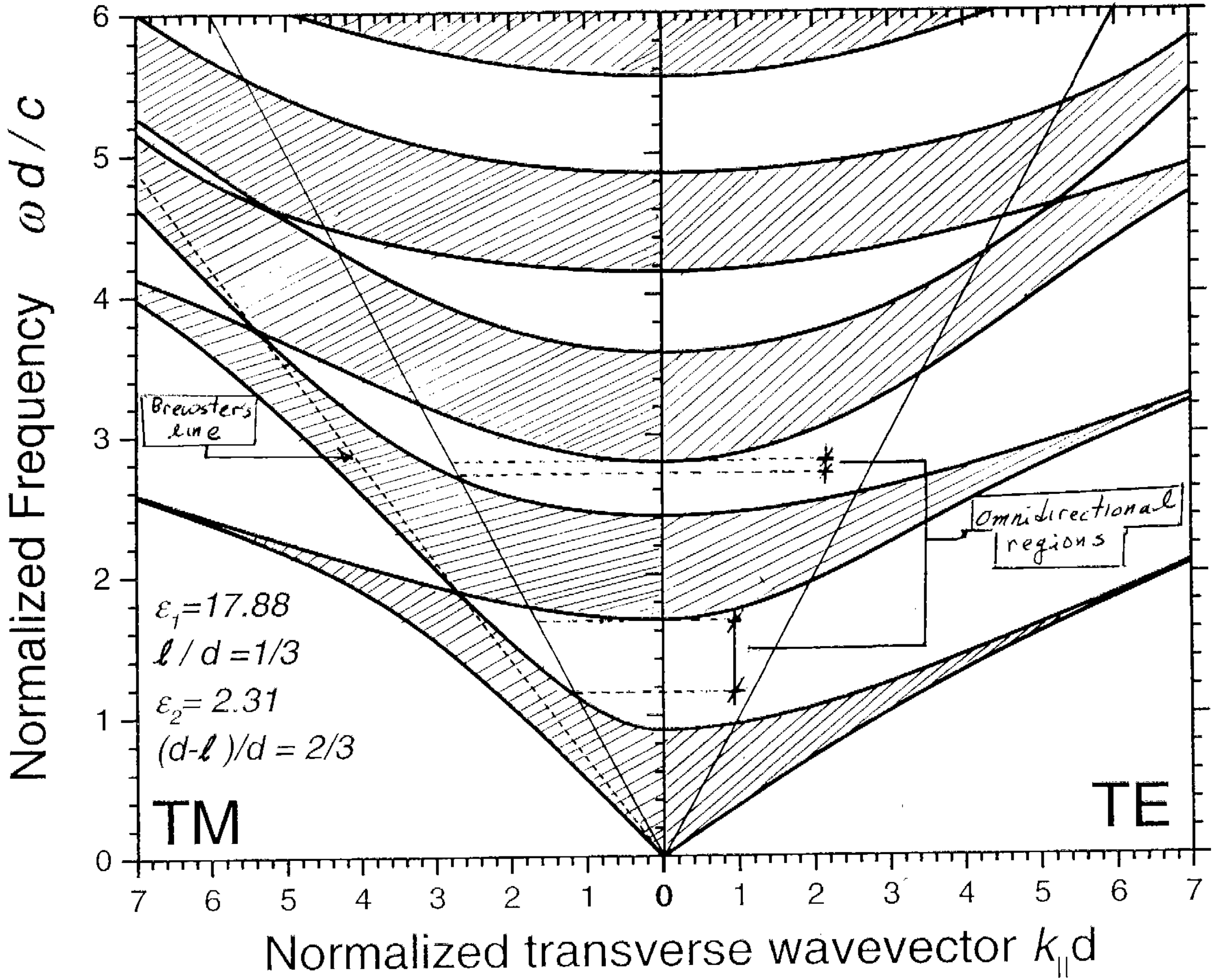
For TM

$$\rho_2 = \frac{k_1 \epsilon_2}{k_2 \epsilon_1}$$

$$\cos k_{B2} d = \cos[k_1 \ell] \cos[k_2 (d-\ell)] - \frac{1}{2}\left[\frac{k_1 \epsilon_2}{k_2 \epsilon_1} + \frac{k_2 \epsilon_1}{k_1 \epsilon_2}\right] \sin[k_1 \ell] \sin[k_2 (d-\ell)]$$

TM

2π Band Diagram for TE and TM modes



3

The objective of this question was that you could realize the origin of the allowed and forbidden bands. Once this could be understood, the answer for the Brewster effect was straight forward.

The origin of the allowed and forbidden bands are Bragg diffraction.

Therefore when $\cos k_{3z} d = \pm 1$, Bragg condition is fulfill and the energy can not travel because of "standing waves".
As you notice all the band edges corresponds to $\cos k_{Bz} d = \pm 1$.

Since for the Brewster effect can not be a reflected wave on the interfaces, then it can not exist Bragg reflection, therefore the bands joint.

To get the condition, easily we set for the TM modes in the boundary conditions $B^{nj} = 0$ it immediatly gives that for non trivial solutions of A^{nj} this must be true

$$\frac{k_1 \epsilon_2}{k_2 \epsilon_1} = 1$$

$$k_1 \epsilon_2 = k_2 \epsilon_1$$

$$\left(\frac{\omega^2}{c^2}\epsilon_1 - k_\parallel^2\right)\epsilon_2^2 = \left(\frac{\omega^2}{c^2}\epsilon_2 - k_\parallel^2\right)\epsilon_1^2$$

$$\frac{\omega^2}{c^2}\epsilon_2^2\epsilon_1 - \epsilon_2\epsilon_1^2 = + k_\parallel^2(\epsilon_2^2 - \epsilon_1^2)$$

$$\frac{\omega^2}{c^2}\epsilon_2\epsilon_1(\epsilon_2 - \epsilon_1) = k_\parallel^2(\epsilon_2+\epsilon_1)(\epsilon_2-\epsilon_1)$$

$$\frac{\omega^2}{c^2} = k_\parallel^2 \frac{\epsilon_2+\epsilon_1}{\epsilon_2\epsilon_1}$$

$$\frac{\omega}{c} = \sqrt{\frac{1}{\epsilon_2}+\frac{1}{\epsilon_1}}\; k_\parallel$$

$$\boxed{\left(\frac{\omega d}{c}\right) = \sqrt{\frac{1}{\epsilon_2}+\frac{1}{\epsilon_1}}\,\left[k_\parallel d\right]}$$

Therefore in our band diagram the Brewster effect is the line with slope $\sqrt{1/\epsilon_2 + 1/\epsilon_1}$. In this line is where the bands joint together.

4 | Omnidirectionality.

We draw in the $\frac{\omega d}{c}$ vs $k_{\parallel} d$ diagram the "light line"
e.g. for vacuum

$$\frac{\omega d}{c} = k_{\parallel} d$$

In our graph is straight line with slope equal to one. We say that the photonic crystal is omnidirectional in a region of frequencies $\mathbb{I}$

$$\mathbb{I} = \left[\frac{\omega_0 d}{c}, \frac{\omega_1 d}{c}\right] \qquad \omega_1 > \omega_0$$

if $\mathbb{I}$ corresponds to a forbidden region of light propagation in which $\frac{\omega d}{c} > k_{\parallel} d$.

Particularly, for our graph, there exist two region inside the two first forbidden bands [out of 4 shown] that the photonic crystal is omnidirectional. See graph