1.1

$$\vec{\mathcal{E}}(\vec{r},t) = \mathrm{Re}\{E(\vec{r})e^{-i\omega_0 t}\}$$

$$\vec{\mathcal{E}}(\vec{r},t) = \{E(\vec{r})e^{-i\omega_0 t} + E^*(\vec{r})e^{+i\omega_0 t}\}\cdot\frac{1}{2}$$

$$\hat{E}(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{\mathcal{E}}(\vec{r},t)e^{-i\omega t}dt$$

$$\hat{E}(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{\vec{E}(\vec{r})}{2}e^{-i\omega_0 t} + \frac{E^*(\vec{r})}{2}e^{i\omega_0 t}\right\}e^{-i\omega t}dt$$

$$\hat{E}(\vec{r},\omega) = \frac{1}{2\sqrt{2\pi}}\left\{E(\vec{r})\int_{-\infty}^{\infty}e^{-i(\omega_0+\omega)t}dt + E^*(\vec{r})\int_{-\infty}^{\infty}e^{-i(\omega-\omega_0)t}dt\right\}$$

$$\hat{E}(\vec{r},\omega) = \frac{1}{2\sqrt{2\pi}}\left\{E(\vec{r})\,2\pi\delta(\omega+\omega_0) + E^*(r)\,2\pi\,\delta(\omega-\omega_0)\right\}$$

$$\hat{E}(\vec{r},\omega) = \sqrt{\frac{\pi}{2}}\left\{E(\vec{r})\,\delta(\omega+\omega_0) + E^*(\vec{r})\,\delta(\omega-\omega_0)\right\}$$

1.2

$$\vec{\mathcal{E}}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{E}(r,\omega)e^{-i\omega t}d\omega \qquad [1]$$

since $\vec{\mathcal{E}}(\vec{r},t)$ is real then

$$\vec{\mathcal{E}}^*(\vec{r},t) = \vec{\mathcal{E}}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{E}^*(r,\omega)e^{-i\omega t}d\omega \qquad [2]$$

$$\vec{\mathcal{E}}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{E}^*(\vec{r},\omega)e^{-i\omega t}d\omega \qquad [3]$$

doing the change of variable in the above equation $\omega'=-\omega$

$$\vec{\mathcal{E}}(\vec{r},t) = \frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty}\hat{E}^*(\vec{r},-\omega')e^{i\omega' t}d\omega' \qquad [4]$$

$$\vec{\mathcal{E}}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{E}^*(\vec{r},-\omega')e^{i\omega' t}d\omega' \qquad [5]$$

Then comparing eq [5] and eq [1] we obtain that

$$\hat{E}^*(\vec{r},-\omega) = \hat{E}(r,\omega)$$

2)
$$\tilde{\vec{E}}(\vec{r},t) = Re\{\vec{\mathcal{E}}(\vec{r},t)e^{i[\vec{k}\cdot\vec{r}-\omega t]}\} \quad [1]$$

$$\tilde{\vec{E}}(\vec{r},t) = \frac{1}{2}\{\vec{\mathcal{E}}(\vec{r},t)e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{\mathcal{E}}^*(\vec{r},t)e^{-i(\vec{k}\cdot\vec{r}+\omega t)}\} \quad [2]$$

Here we can work with only one part the other will be the conjugate of the other part.

The wave equation in empty space is:

$$\nabla^2\vec{E} = +\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} \quad [3]$$

defining
$$\vec{E}(\vec{r},t) = \vec{\mathcal{E}}(\vec{r},t)e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad [4]$$

the
$$\nabla^2\vec{E} = \nabla^2\{\vec{\mathcal{E}}(\vec{r},t)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\}$$
$$\nabla^2\vec{E} = \nabla^2\{\mathcal{E}_x(\vec{r},t)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\}\hat{x} + \nabla^2\{\mathcal{E}_y(\vec{r},t)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\}\hat{y}$$
$$+ \nabla^2\{\mathcal{E}_z(\vec{r},t)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\}\hat{z}$$

$$\nabla^2\vec{E} = \sum_m\sum_j\left[\frac{\partial^2\mathcal{E}_m}{\partial x_j^2} + 2ik_j\frac{\partial\mathcal{E}_m}{\partial x_j} - k_j^2\mathcal{E}_m\right]\hat{x}_m e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

For $m=1,2,3$, $j=1,2,3$ then
$$\nabla^2\vec{E} = \left(\nabla^2\vec{\mathcal{E}} - k^2\vec{\mathcal{E}} + 2i(\nabla\vec{\mathcal{E}})\cdot\vec{k}\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad [5]$$

Here $\nabla\vec{\mathcal{E}}$ is a dyadic defined as:

$$\nabla\vec{\mathcal{E}} \equiv \nabla\mathcal{E}_x\hat{x} + \nabla\mathcal{E}_y\hat{y} + \nabla\mathcal{E}_z\hat{z} \quad [6]$$

Now
$$\frac{\partial^2\vec{E}}{\partial t^2} = \left(\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} - 2i\omega\frac{\partial\vec{\mathcal{E}}}{\partial t} - \omega^2\vec{\mathcal{E}}\right)e^{i\vec{k}\cdot\vec{r}-\omega t} \quad [7]$$

Substituting eqs [7] and [5] into eq. [3]

$$\nabla^2\vec{\mathcal{E}} - k^2\vec{\mathcal{E}} + 2i\nabla\vec{\mathcal{E}}\cdot\vec{k} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} - \frac{2i\omega}{c^2}\frac{\partial\vec{\mathcal{E}}}{\partial t} - \frac{\omega^2}{c^2}\vec{\mathcal{E}}$$

since in free space
$$k^2 = \frac{\omega^2}{c^2}$$

then the above equation reduces to
$$\boxed{\nabla^2\vec{\mathcal{E}} + 2i\nabla\vec{\mathcal{E}}\cdot\vec{k} = \frac{1}{c^2}\frac{\partial^2\vec{\mathcal{E}}}{\partial t^2} - \frac{2i\omega}{c^2}\frac{\partial\vec{\mathcal{E}}}{\partial t}} \quad [8]$$

also $E(\vec{r},t)$ must satisfy

$$\nabla\cdot\vec{E}(\vec{r},t) = 0$$
$$\nabla\cdot\{\vec{\mathcal{E}}(\vec{r},t)e^{i\vec{k}\cdot\vec{r}-\omega t}\} = \sum_m\left(\frac{\partial\mathcal{E}_m}{\partial m} + ik_m\mathcal{E}_m\right)e^{i(\vec{k}\cdot\vec{r}-\omega t)} = 0$$

then
$$\boxed{\nabla\cdot\vec{\mathcal{E}}(\vec{r},t) + i\vec{k}\cdot\vec{\mathcal{E}} = 0} \quad [9]$$

Comments:

Usually for optical wavelengths, and since $\mathcal{E}(\vec{r},t)$ is spatially and temporaly small variant, i.e.

$$\left| 2i \nabla \vec{\mathcal{E}} \cdot \vec{k} \right| \gg \left| \nabla^2 \vec{\mathcal{E}} \right|$$

and $$\left| \frac{1}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t} \right| \ll \left| \frac{2i\omega}{c} \frac{\partial \vec{\mathcal{E}}}{\partial t} \right|$$

so eq [8] can be approximated as

$$2i \nabla \vec{\mathcal{E}} \cdot \vec{k} = -\frac{2i\omega}{c^2} \frac{\partial \vec{\mathcal{E}}}{\partial t}$$

this is known as the "slow variant envelope approximation".

The divergence $\nabla \cdot \vec{E} = 0$ {for this case} is a very important condition that must be satisfied.

2.2)

The intensity is given by:

$$I(t) = \left| \vec{\mathcal{E}}(\vec{r},t)\, e^{i\vec{k}\cdot r - \omega_0 t} \right|^2 \qquad [1]$$

at $r=0$ we obtain

$$I(t) = |\vec{\mathcal{E}}(0,t)|^2 = \vec{\mathcal{E}}(0,t)\cdot\vec{\mathcal{E}}^*(0,t)\, e^{i\omega_0 t} e^{-i\omega_0 t} = \vec{\mathcal{E}}(0,t)\cdot\vec{\mathcal{E}}^*(0,t) \qquad [2]$$

then for

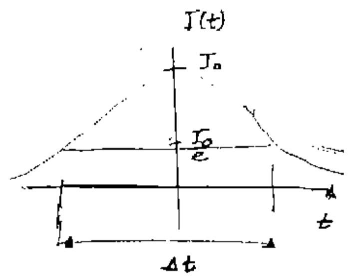
$$\vec{\mathcal{E}}(0,t) = \vec{E}_0 e^{-t^2/2\tau^2}$$

$$I(t) = \vec{E}_0\cdot\vec{E}_0^*\left(e^{-t^2/2\tau^2}\right)^2 = |E_0|^2 e^{-t^2/\tau^2} \qquad [3]$$

we find Δt at $\frac{1}{e}$ so

$$I\left(\frac{\Delta t}{2}\right) = \frac{I_0}{e} \Rightarrow \frac{1}{e} = e^{-t/\tau^2} \Rightarrow -1 = \frac{-(\Delta t)^2}{4\tau^2}$$

therefore $\boxed{\Delta t = 2\tau}$ [4]



Taking the Fourier transform of $\mathcal{E}(0,t)e^{-i\omega_0 t}$. See comments at the end since this is an approximation. Therefore:

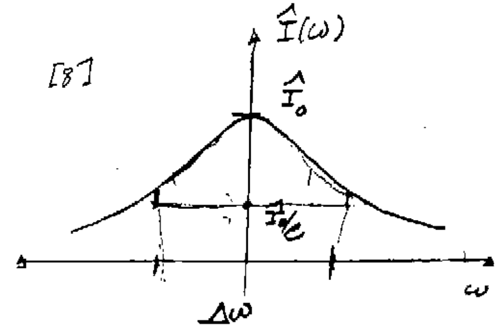
$$\hat{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}_0 e^{-t^2/2\tau^2} e^{-i\omega_0 t} e^{i\omega t}\, dt \qquad [6]$$

$$\hat{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}_0 e^{-t^2/2\tau} e^{i(\omega-\omega_0)t}\, dt \qquad [7]$$

$$\hat{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}_0 e^{-\frac{t^2}{2\tau} + i(\omega_0-\omega)t}\, dt$$

$$\hat{E}(\omega) = \frac{1}{\sqrt{2\pi}}\vec{E}_0 e^{-\frac{(\omega-\omega_0)^2\tau^2}{2}}\sqrt{\pi}\sqrt{2}\,\tau$$

$$\hat{E}(\omega) = \vec{E}_0\tau e^{-\frac{(\omega-\omega_0)^2\tau^2}{2}} \qquad [8]$$



Then

$$\hat{I}(\omega) = |\hat{E}(\omega)|^2$$

$$\hat{I}(\omega) = |\vec{E}_0|^2\tau^2 e^{-(\omega-\omega_0)^2\tau^2} \qquad [9]$$

then

$$I\left(\frac{\Delta\omega}{2}+\omega_0\right) = \frac{I(\omega_0)}{e} = |\vec{E}_0|\tau^2 e^{-\frac{(\Delta\omega)^2}{4}\tau^2}$$

$$-1 = -\frac{\Delta\omega^2}{4}\tau^2 \Rightarrow \boxed{\Delta\omega = \frac{2}{\tau}} \qquad [10]$$

From eqs. [9] and [10]

$$\Delta\omega\Delta\tau = \frac{2}{\tau}\cdot 2\tau = 4$$

$$\boxed{\Delta\omega\Delta\tau = 4} \qquad [11]$$

2.3 with Chirp

$$\vec{\mathcal{E}}(0,t)e^{-i\omega_0 t} = \vec{E}_0 e^{-t^2/2\tau^2} e^{i\omega_0 t} e^{-i\beta t^2} \qquad [1]$$

$$\mathcal{I}(t) = \left|\vec{\mathcal{E}}(0,t)e^{-i\omega_0 t}\right|^2 = |\vec{E}_0|^2 e^{-t^2/\tau^2} \qquad [2]$$

Therefore the pulse temporal width is the same as in problem 2.2, namely

$$\Delta t = 2\tau \qquad [3]$$

Taking the Fourier transform of [3] we obtain

$$\vec{E}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}_0\, e^{-\left[\frac{1}{2\tau^2}+\beta i\right]t^2} e^{i(-\omega_0+\omega)t}\,dt \qquad [4]$$

$$\hat{E}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{p}\, e^{-\frac{(\omega-\omega_0)^2}{4p^2}} \qquad [5]$$

where

$$p^2 \equiv \frac{1}{2\tau^2} + \beta i = \sqrt{\frac{1}{4\tau^4}+\beta^2}\; e^{i\gamma} \qquad [6]$$

where $\gamma \equiv$ ~~[illegible]~~ $\operatorname{arctg}[\beta 2\tau^2]$ [7]

then $p = \left[\frac{1}{4\tau^4}+\beta^2\right]^{\frac{1}{4}} e^{i\frac{\gamma}{2}}$ [8]

Then using eqs [6-8] of [5] becomes

$$\hat{E}(\omega) = \frac{1}{\sqrt{2}}\frac{\vec{E}_0\, e^{-i\frac{\gamma}{2}}}{\left[\frac{1}{4\tau^4}+\beta^2\right]^{1/4}}\, Exp\left[\frac{-(\omega-\omega_0)^2}{4\left(\frac{1}{4\tau^4}+\beta^2\right)}\left[\frac{1}{2\tau^2}-\beta i\right]\right]$$

$$\vec{E}(\omega) = \frac{1}{\sqrt{2}}\frac{\vec{E}_0\, e^{-i\frac{\gamma}{2}}}{\left[\frac{1}{4\tau^4}+\beta^2\right]^{1/4}}\, Exp\left[\frac{-(\omega-\omega_0)^2(\tau^2)}{(1+4\tau^4\beta^2)2}\right] Exp\left[+i\delta\right] \qquad [9]$$

where

$$\delta \equiv \frac{(\omega-\omega_0)^2\beta}{\left[\frac{1}{4\tau^4}+\beta^2\right]} \qquad [10]$$

Then

$$\hat{\mathcal{I}}(\omega) = \frac{|\vec{E}_0|^2}{\sqrt{\frac{1}{4\tau^4}+\beta^2}}\, e^{-\frac{(\omega-\omega_0)^2\tau^2}{(1+4\tau^4\beta^2)}}$$

we now find the width

$$\hat{I}\left(\frac{\Delta\omega}{2}+\omega_0\right) = \frac{\hat{\mathcal{I}}(\omega_0)}{e} \implies \frac{1}{e} = e^{-\frac{\Delta\omega^2\tau^2}{(1+4\tau^4\beta^2)4}}$$

$$\Delta\omega^2 = \frac{4}{\tau^2} + 16\tau^2\beta^2$$

$$\Delta\omega = \frac{2}{\tau}\sqrt{1+4\tau^4\beta^2} \qquad [11]$$

Then from eqs [14] and eq [3] $\Delta\omega\,\Delta t$ is

$$\boxed{\Delta\omega\,\Delta t = 4\sqrt{1 + 4\tau^4\beta^2} \qquad [17]}$$

$\Delta\omega\,\Delta t$ minimizes when $\beta = 0$, that is no chirp

2.4

$$\hat{P}(\omega) = \varepsilon_0 \chi(\omega)\hat{E}(\omega)$$

$$\hat{P}(\omega) = \frac{\varepsilon_0\,\omega_p^2}{\omega_n^2}\left(1 + \frac{\omega^2}{\omega_n^2}\right)\hat{E}(\omega)$$

Therefore

$$\vec{P}(\vec{r}=0,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\varepsilon_0\,\omega_p^2}{\omega_n^2}\left\{1 + \frac{\omega^2}{\omega_n^2}\right\}\hat{E}(\omega)e^{-i\omega t}\,d\omega$$

$$\vec{P}(\vec{r}=0,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\varepsilon_0\,\omega_p^2}{\omega_n^2}\hat{E}(\omega)e^{-i\omega t}\,d\omega + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\varepsilon_0\,\omega_p^2}{\omega_n^2}\frac{\omega^2}{\omega_n^2}\hat{E}(\omega)e^{-i\omega t}\,d\omega$$

$$\vec{P}(\vec{r}=0,t) = \frac{\varepsilon_0\,\omega_p^2}{\omega_n^2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{E}(\omega)e^{-i\omega t}\,d\omega\right\} + \frac{\varepsilon_0\,\omega_p^2}{\omega_n^4}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{E}(\omega)\,(+i\omega)^2}{i^2}e^{-i\omega t}\,d\omega\right\}$$

$$P(\vec{r},t) = \frac{\varepsilon_0\,\omega_p^2}{\omega_n^2}\,\mathcal{E}(0,t)e^{-i\omega_0 t} - \frac{\partial^2}{\partial t^2}\left\{\mathcal{E}(0,t)e^{-i\omega_0 t}\right\}\frac{\varepsilon_0\,\omega_p^2}{\omega_n^4}$$

$$P(\vec{r},t) = \frac{\varepsilon_0\,\omega_p^2}{\omega_n^2}\,\mathcal{E}(0,t)e^{-i\omega t} - \frac{\varepsilon_0\,\omega_p^2}{\omega_n^4}\left\{\frac{d^2\mathcal{E}(0,t)}{dt^2} - 2i\omega_0\frac{d\mathcal{E}(0,t)}{dt} - \omega_0^2\mathcal{E}(0,t)\right\}e^{-i\omega_0 t}$$

$$\vec{P}(\vec{r},t) = \frac{\epsilon_0 \omega_p^2}{\omega_\eta^2} \mathcal{E}(0,t) e^{-i\omega_0 t} - \frac{\epsilon_0 \omega_p^2}{\omega_\eta^4} \left[-\frac{1}{\tau^2} + \frac{1}{\tau^4} t^2 + \frac{2i\omega_0 t}{\tau^2} - \omega_0^2 \right] \times$$
$$\vec{\mathcal{E}}(0,t) e^{-i\omega_0 t}$$

$$\boxed{\vec{P}(\vec{r},t) = \left\{ \frac{\epsilon_0 \omega_p^2}{\omega_\eta^2} + \frac{\epsilon_0 \omega_p^2}{\omega_\eta^4} \left[\left(\frac{1}{\tau^2} + \omega_0^2 \right) - \frac{1}{\tau^4} t^2 - \frac{2i\omega_0 t}{\tau^2} \right] \right\} \mathcal{E}(\vec{r},t) e^{-i\omega_0 t}}$$

Comments.

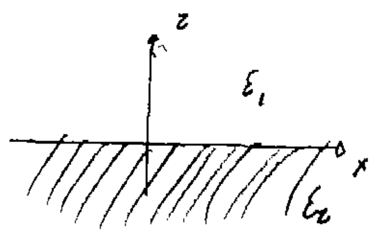
Problems 2.2–2.4, The Fourier transforms involve the complex part only. This approach is an approximation. Think about how reasonable is to do that. Set typical values for ω_0 in the optical regime, typical τ for pulses originated in lock-in lasers. Do we have strictly Gaussian pulses in real life (they extend from $-\infty < t < \infty$)?

3]

-i] $\vec{E}^n(\vec{r}) = \left(E_{nx}\,\hat{n}_x + E_{nz}\,\hat{n}_z\right) e^{i(k_{nx}x + k_{nz}z)}$ [1]

Gauss Law

$\nabla \cdot E(\vec{r}) = 0$



From eq [1]

$$i\left(k_{nx} E_{nx} + k_{nz} E_{nz}\right) e^{i(k_{nx}x + k_{nz}z)} = 0$$

then

$$\boxed{k_{nx} E_{nx} + k_{nz} E_{nz} = 0 \qquad n = 1,2}$$ [2]

-ii]

- Boundary condition the tangential components of the electric field are continuos therefore

at $z=0$

$$\vec{E}_1 \cdot \hat{x} = \vec{E}_2 \cdot \hat{x}$$

$$E_{1x} e^{ik_{x1}x} = E_{2x} e^{ik_{x2}x}$$ [3]

Since eq [3] must be fullfill for all values of x then

$$k_{x_1} = k_{x_2} = k_x$$ [4]

and

$$E_{1x} = E_{2x} = E_x$$ [5]

- Boundary condition: The normal components of $\vec{D}$ are continuos, then

$$\hat{z} \cdot \vec{D}_1 = \vec{D}_2 \cdot \hat{z}$$ [6]

$$\vec{D}_1(\vec{r}) = \epsilon_1 \epsilon_0 \vec{E}_1(\vec{r})$$ [7]

$$\vec{D}_2(\vec{r}) = \epsilon_2 \epsilon_0 \vec{E}_2(\vec{r})$$ [8]

From eqs [7] and [8] using eq [6] and eq [1]

$$\epsilon_1 E_{z1} = \epsilon_2 E_{z2}$$

So we have the following eqs.

$$k_x E_x + k_{z1} E_{z1} = 0$$ [9]

$$k_x E_x + k_{z2} E_{z2} = 0$$ [10]

$$\epsilon_1 E_{z1} = \epsilon_2 E_{z2}$$ [11]

From eqs [9] and [10]

$$k_{z1} E_{z1} = k_{z2} E_{z2}$$ [12]

and from eq [11] and using eq [12]

$$\frac{\epsilon_2}{\epsilon_1} k_{z1} = k_{z2}$$ [13]

From the wave equation:

$$k_{z1} = \sqrt{\frac{\omega^2}{c^2}\epsilon_1 - k_x^2} \qquad k_{z2} = \sqrt{\frac{\omega^2}{c^2}\epsilon_2 - k_x^2}$$

Substituting the expressions of k_{z_1} and k_{z_2} into eq [13]

$$\frac{\epsilon_2}{\epsilon_1}\sqrt{\frac{\omega^2}{c^2}\epsilon_1 - k_x^2} = \sqrt{\frac{\omega^2}{c^2}\epsilon_2 - k_x^2}$$

$$\frac{\epsilon_2^2}{\epsilon_1^2}\left(\frac{\omega^2}{c^2}\epsilon_1 - k_x^2\right) = \left(\frac{\omega^2}{c^2}\epsilon_2 - k_x^2\right)$$

$$\frac{\omega^2}{c^2}\left[\frac{\epsilon_2^2}{\epsilon_1} - \epsilon_2\right] = k_x^2\left[\frac{\epsilon_2^2}{\epsilon_1^2} - 1\right]$$

$$\frac{\omega^2}{c^2}\frac{\epsilon_2(\epsilon_2-\epsilon_1)}{\epsilon_1} = k_x^2\frac{\epsilon_2^2-\epsilon_1^2}{\epsilon_1^2}$$

$$\frac{\omega^2}{c^2}\epsilon_2 = k_x^2\frac{(\epsilon_2+\epsilon_1)}{\epsilon_1}$$

$$k_x^2 = \frac{\omega^2}{c^2}\frac{\epsilon_2\epsilon_1}{\epsilon_2+\epsilon_1}$$

$$\boxed{k_x = \frac{\omega}{c}\sqrt{\frac{\epsilon_2\epsilon_1}{\epsilon_2+\epsilon_1}}} \qquad [14]$$

iii] Derive the magnetic and show that fullfills the boundary conditions.

$$\nabla\times\vec{E}(\vec{r}) = +i\omega\vec{B}(\vec{r}) \qquad [15]$$

$$\vec{B}^{(n)}(\vec{r}) = \frac{-i}{\omega}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z}\\ \tilde{E}_x^{(n)}(\vec{r}) & 0 & \tilde{E}_z^{(n)}(\vec{r})\end{vmatrix} = \frac{+i}{\omega}\hat{y}\left[\frac{\partial\tilde{E}_z^{(n)}(\vec{r})}{\partial x} - \frac{\partial\tilde{E}_x^{(n)}(\vec{r})}{\partial z}\right] \qquad [16]$$

Here we have define

$$\tilde{E}_x^{(n)}(\vec{r}) \equiv E_x e^{i(k_x x + k_{nz} z)} \quad \text{and} \quad \tilde{E}_y^{(n)}(\vec{r}) = E_{zn} e^{i(k_x x + k_{nz} z)} \qquad [17]$$

Therefore

$$\vec{B}^{(n)}(\vec{r}) = \frac{\hat{y}}{\omega}\left[k_x E_{zn} - k_{zn}E_x\right]e^{i(k_x x + k_z z)} \qquad [18]$$

Since the medium is <u>non</u> magnetic the tangential components are continuos, that is

$$\hat{y}\cdot\vec{B}^{(1)}(x, z=0) = \vec{B}^{(2)}(x, z=0)\cdot\hat{y} \qquad [19]$$

Now we must show that the above equation is satisfied by using the conditions derived before, thus

From eqs [19] and [18]

$$k_x E_{z_1} - k_{z_1}E_x = k_x E_{z_2} - k_{z_2}E_x \qquad [19]$$

Here there are many options to follow

using eqs [9] and [10] into eq [19]

$$-\frac{k_x^2 E_x}{k_{z_1}} - k_{z_1} E_x = -\frac{k_x^2 E_x}{k_{z_2}} - k_{z_2} E_x$$

$$E_x \left\{ -\frac{(k_x^2+k_{z_1}^2)}{k_{z_1}} + \frac{(k_x^2+k_{z_2}^2)}{k_{z_2}} \right\} = 0$$

$$k_{z_n}^2 + k_x^2 = \frac{\omega^2}{c^2}\epsilon_n \qquad n=1,2$$

then

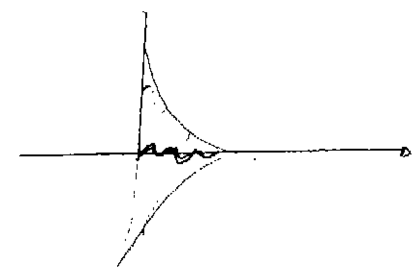
$$-\frac{\epsilon_1}{k_{z_1}} + \frac{\epsilon_2}{k_{z_2}} = 0$$

$$k_{z_2} = k_{z_1}\frac{\epsilon_2}{\epsilon_1} \qquad [21]$$

This condition is precisely the eq [13], therefore we show that the boundary condition for the $\vec{B}$ field is fullfill.

-iv] we are looking for

k_x real, k_{z_1}, k_{z_2} purely imaginary so



therefore assuming $\epsilon_1 > 0$ then

ϵ_1, ϵ_2 are real

- k_x is real if:
  - $\epsilon_1 > 0$, $\epsilon_2 > 0$

  or
  - $\epsilon_1 > 0$ $\epsilon_2 < 0$ $|\epsilon_2| > |\epsilon_1|$

- k_{z_n} is purely imaginary

$$\frac{\omega^2}{c^2}\epsilon_1 < k_x^2$$

From dispertion relation

$$\cancel{\frac{\omega^2}{c^2}}\epsilon_1 < \cancel{\frac{\omega^2}{c^2}}\frac{\epsilon_2\epsilon_1}{\epsilon_2+\epsilon_1}$$

$$1 < \frac{\epsilon_2}{\epsilon_2+\epsilon_1} \Rightarrow 1 < \frac{1}{1+\frac{\epsilon_1}{\epsilon_2}}$$

if $\epsilon_1 > 0$

$\left|\frac{\epsilon_1}{\epsilon_2}\right| < 1$ and $\epsilon_1 > 0$ and $\epsilon_2 < 0$

- but now

$$k_{z2} = \frac{\epsilon_1}{\epsilon_2} k_{z1}$$

if $\left|\frac{\epsilon_1}{\epsilon_2}\right| < 1$ and $(\epsilon_1 > 0, \epsilon_2 < 0)$

k_{z1} is purely imaginary,

since $\frac{\epsilon_1}{\epsilon_2}$ are real the k_{z2} is purely imaginary

Thus for k_x real and the field vanishing at $z \to \pm\infty$

$$\left|\frac{\epsilon_1}{\epsilon_2}\right| < 1 \quad \text{and} \quad (\epsilon_1 > 0, \epsilon_2 < 0)$$

This is a plasmon

V]

$$\epsilon_2(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \qquad [22]$$

From the above expression we obtain a plasmon when

$\omega < \omega_p$ and $\frac{|\epsilon_1|}{|\epsilon_2|} < 1$

then

$$k_x^2 = \frac{\omega^2}{c^2}\frac{\epsilon_2\epsilon_1}{\epsilon_1+\epsilon_2} = \frac{\omega^2}{c^2}\frac{\epsilon_1\left[1 - \frac{\omega_p^2}{\omega^2}\right]}{\epsilon_1 + 1 - \frac{\omega_p^2}{\omega^2}}$$

Let define

$$q \equiv \frac{\omega}{\omega_p} \quad \text{and} \quad k \equiv \frac{c k_x}{\omega_p}$$

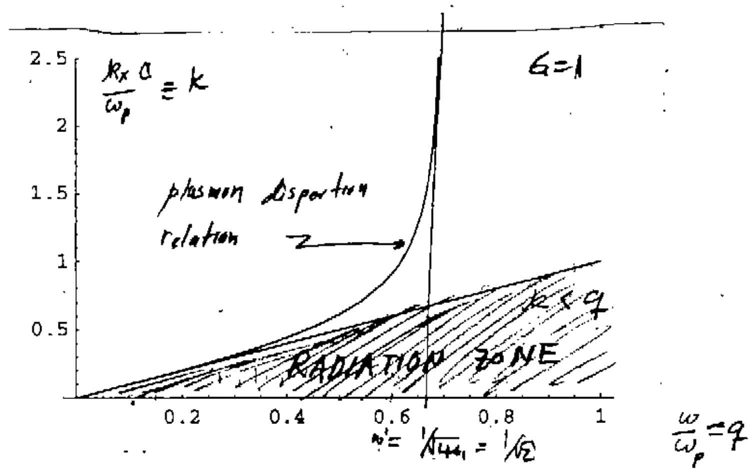
then

$$k^2 = \frac{q^2\epsilon_1\left[1 - \frac{1}{q^2}\right]}{\epsilon_1 + 1 - \frac{1}{q^2}}$$

$$k^2 = \frac{q^4\epsilon_1\left[1 - 1/q^2\right]}{(\epsilon_1+1)q^2 - 1}$$

$$\boxed{k = \sqrt{\frac{\epsilon_1\left[q^4 - q^2\right]}{(\epsilon_1+1)q^2 - 1}}} \qquad [23]$$

plotting eq [23] $\frac{k_x c}{\omega_p}$ vs $\frac{\omega}{\omega_p}$ for $\epsilon_1 = 1.00$



we notice that a plasmon can not be generated by striking light at the interface, since the radiation zone is outside the dispertion curve. when $\omega \to \frac{1}{\sqrt{1+\epsilon_1}}$, in our case we set $\epsilon_1 = 1$, $\frac{k_x c}{\omega_p}$ is infinite.