# OPT 462: Solutions to Homework Set 12

## 1 Emission from a truncated waveguide

Consider the fields emerging from a truncated quadratic hollow metal waveguide with ideally conducting walls. The side length $a_o$ is chosen in a way that only the lowest order $TE_{10}$ mode polarized in $x$-direction is supported. The field distribution of this mode is given by

$$E_x = -i\omega\mu_o\mu\,\frac{a_o}{\pi}\,E_{01}\cos\left(\frac{\pi}{a_o}y\right)\,\mathrm{rect}\left[\frac{x}{a_o}\right]\,\mathrm{rect}\left[\frac{y}{a_o}\right]\,\mathrm{e}^{ik_z z} \quad (1)$$

$$E_y = 0 \quad (2)$$

$$E_z = 0$$

Assume that the fields are not influenced by the edges of the truncated side walls.

- Calculate the Fourier spectrum of the electric field in the exit plane ($z=0$).
- Determine the field for arbitrary distance $z$.
- Calculate and plot the corresponding farfield.

Calculate Fourier spectrum at $z=0$:

$$\begin{aligned}
\hat{E}_x(k_x,k_y;0) &= -i\omega\mu_o\mu\,\frac{a_o}{\pi}\,E_{01}\iint\limits_{-\infty}^{\infty}\cos\left(\frac{\pi}{a_o}\,y\right)\,\mathrm{rect}\left[\frac{x}{a_o}\right]\,\mathrm{e}^{-i(k_x x+k_y y)}\,dx\,dy \\
&= -i\omega\mu_o\mu\,\frac{a_o}{\pi}\,E_{01}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-ik_x x}\,\mathrm{rect}\left[\frac{x}{a_o}\right]dx\int\limits_{-\infty}^{\infty}\cos\left(\frac{\pi}{a_o}y\right)\,\mathrm{rect}\left[\frac{y}{a_o}\right]\,\mathrm{e}^{-ik_y y}\,dy \\
&= -i\omega\mu_o\mu\,\frac{2a_o}{k_x\pi}\,E_{01}\sin\left(\frac{a_o}{2}k_x\right)\left[\frac{\sin(\frac{\pi}{2}-\frac{a_o}{2}k_y)}{\frac{\pi}{a_o}-k_y}+\frac{\sin(\frac{\pi}{2}+\frac{a_o}{2}k_y)}{\frac{\pi}{a_o}+k_y}\right] \\
&= -4i\omega\mu_o\mu\,E_{01}\frac{\sin(\frac{a_o}{2}k_x)\,\cos(\frac{a_o}{2}k_y)}{k_x\left[(\pi/a_o)^2-k_y^2\right]}
\end{aligned}$$

Determine field for $z>0$:

$$E_x(\mathbf{r}) = \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\hat{E}_x(k_x,k_y;0)\,\mathrm{e}^{i\left[k_x x+k_y y+\sqrt{k^2-k_x^2-k_y^2}\,z\right]}\,dk_x\,dk_y$$

with the expression for $\hat{E}_x(k_x, k_y; 0)$ from above.

Calculate Farfields:

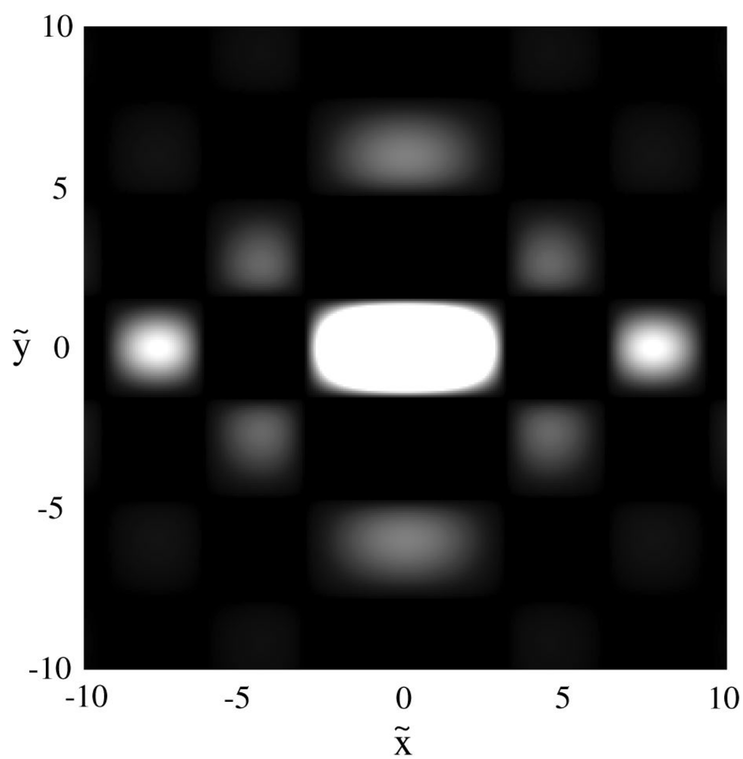
$$\hat{E}_x(s_x, s_y, s_z) = -\frac{4i\omega\mu_o\mu}{k} E_{01} \frac{\sin\left(\frac{ka_o}{2}s_x\right)\cos\left(\frac{ka_o}{2}s_y\right)}{s_x\left[(\pi/a_o)^2 - k^2 s_y^2\right]}$$

$$\mathbf{E}_\infty\,(s_x, s_y, s_z) = -iks_z\hat{E}_x(s_x, s_y, s_z)\,\frac{\mathrm{e}^{ikr}}{r}\,\mathbf{n}_x$$

Expressed in normalized cartesian coordinates using $s_x = \tilde{x}/r$, $s_y = \tilde{y}/r$, and $s_z = \tilde{z}/r$, where $\tilde{x} = ka_o x/2r$, $\tilde{y} = ka_o y/2r$, $\tilde{z} = ka_o z/2r$:

$$\mathbf{E}_\infty\,(x, y, z) = -\omega\mu_o\mu a_o^2\,E_{01}\,\tilde{z}\,\frac{\mathrm{e}^{ikr}}{r}\left[\frac{\sin\tilde{x}\,\cos\tilde{y}}{\tilde{x}\,(\pi^2/4 - \tilde{y}^2)}\right]\mathbf{n}_x$$

The farfield radiation has spherical wavefronts. The radiation pattern is determined by the expression in brackets. The factor $\tilde{z}$ only smoothes the radiation pattern with a factor $\cos\theta$ which makes sure that there is no radiation at right angles to the truncated waveguide. A contourplot of the expression in brackets is shown in the figure blow. Most of the radiation propagates in forward direction but many weak sidelobes are formed. The center lobe is elongated in direction of polarization.



## 2 Farfield of a radiating dipole

Consider a dipole aligned with the z-axis and located at $z=0$. Use the Weyl representation (see notes) to find the angular spectrum representation.

- Derive the farfields and show that they are identical to the familiar formula expressed in spherical coordinates.

The dyadic Green's function $\overleftrightarrow{\mathbf{G}}_o\,(\mathbf{r},\mathbf{r}_o)$ defines the electric field $\mathbf{E}(\mathbf{r})$ of an electric dipole $\mathbf{p}$ located at $\mathbf{r}_o=(x_o,y_o,z_o)$ according to

$$\mathbf{E}(\mathbf{r}) \;=\; \omega^2\mu_o\mu\, \overleftrightarrow{\mathbf{G}}_o(\mathbf{r},\mathbf{r}_o)\,\mathbf{p}\,. \tag{3}$$

The material parameters and the oscillation frequency determine the wavenumber $k$ and its longitudinal component $k_z$. To represent $\overleftrightarrow{\mathbf{G}}_o$ by an angular spectrum we first consider the vector potential $\mathbf{A}$ which satisfies

$$\left[\nabla^2 + k^2\right]\mathbf{A}(\mathbf{r}) \;=\; -\mu_o\mu\,\mathbf{j}(\mathbf{r})\,. \tag{4}$$

Here, $\mathbf{j}$ is the current density of the dipole which reads as

$$\mathbf{j}(\mathbf{r}) \;=\; -i\omega\,\delta(\mathbf{r}-\mathbf{r}_o)\;\mathbf{p}\,. \tag{5}$$

Using the definition of the scalar Green's function $G_o$ we obtain

$$\mathbf{A}(\mathbf{r}) \;=\; \mathbf{p}\,\frac{k^2}{i\omega\varepsilon_o\varepsilon}\,\frac{\mathrm{e}^{ik|\mathbf{r}-\mathbf{r}_o|}}{4\pi|\mathbf{r}-\mathbf{r}_o|}\,. \tag{6}$$

Notice, that the vector potential is polarized in direction of the dipole moment. We now introduce the Weyl identity defined and rewrite the vector potential as

$$\mathbf{A}(\mathbf{r}) \;=\; \mathbf{p}\,\frac{k^2}{8\pi^2\omega\varepsilon_o\varepsilon}\iint\limits_{-\infty}^{\infty}\frac{1}{k_z}\,\mathrm{e}^{i[k_x(x-x_o)+k_y(y-y_o)+k_z|z-z_o|]}\,dk_x\,dk_y\,. \tag{7}$$

Using $\mathbf{E}=i\omega[1+k^{-2}\nabla\nabla\cdot]\mathbf{A}$ it is straightforward to derive the electric field. Similarly, the magnetic field is calculated using $\mathbf{H}=(\mu_o\mu)^{-1}\nabla\times\mathbf{A}$. The resulting expression for $\mathbf{E}$ can be compared with Eq. 3 which allows us to identify the dyadic Green's function as

$$\begin{gathered}\overleftrightarrow{\mathbf{G}}_o(\mathbf{r},\mathbf{r_o}) = \frac{i}{8\pi^2}\iint\limits_{-\infty}^{\infty}\overleftrightarrow{\mathbf{M}}\,\mathrm{e}^{i[k_x(x-x_o)+k_y(y-y_o)+k_z|z-z_o|]}\,dk_x\,dk_y\\[2ex] \overleftrightarrow{\mathbf{M}} = \frac{1}{k^2k_z}\begin{bmatrix} k^2-k_x^2 & -k_xk_y & \mp k_xk_z\\ -k_xk_y & k^2-k_y^2 & \mp k_yk_z\\ \mp k_xk_z & \mp k_yk_z & k^2-k_z^2\end{bmatrix}\end{gathered} \tag{8}$$

Some terms in the matrix $\overleftrightarrow{\mathbf{M}}$ have two different signs. This originates from the absolute value $|z-z_o|$. The upper sign applies for $z>z_o$ and the lower sign for $z<z_o$. Eq. 8 allows us to express

the fields of an arbitrarily oriented dipole in terms of plane waves and evanescent waves.

To calculate the farfield we set $\mathbf{r} = (x_o, y_o, z_o) = \mathbf{0}$ and use

$$\overset{\leftrightarrow}{\mathbf{G}}_\infty\,(s_x, s_y, s_z) = -iks_z\,\hat{\overset{\leftrightarrow}{\mathbf{G}}}_x(s_x, s_y, s_z)\,\frac{\mathrm{e}^{ikr}}{r}\;, \tag{9}$$

From Eq. 8 the spectrum $\hat{\overset{\leftrightarrow}{\mathbf{G}}}$ can be identified as

$$\hat{\overset{\leftrightarrow}{\mathbf{G}}} \;=\; \frac{i}{4\pi}\,\overset{\leftrightarrow}{\mathbf{M}}\;\;. \tag{10}$$

Using a representation in spherical coordinates

$$\begin{aligned} s_x &= k_x/k &&= \cos\phi\,\sin\theta \\ s_y &= k_y/k &&= \sin\phi\,\sin\theta \\ s_z &= k_z/k &&= \cos\theta\;, \end{aligned}$$

we find for $\overset{\leftrightarrow}{\mathbf{G}}_\infty$

$$\overset{\leftrightarrow}{\mathbf{G}}_\infty\,(\mathbf{r}, 0) = \frac{\exp(ikr)}{4\pi r}\begin{bmatrix} (1-\cos^2\!\phi\sin^2\theta) & -\sin\phi\cos\phi\sin^2\theta & -\cos\phi\sin\theta\cos\theta \\ -\sin\phi\cos\phi\sin^2\theta & (1-\sin^2\!\phi\sin^2\theta) & -\sin\phi\sin\theta\cos\theta \\ -\cos\phi\sin\theta\cos\theta & -\sin\phi\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}. \tag{11}$$

The farfield of the electric field $\mathbf{E}_\infty$ is defined by Eq. 3 and the derived farfield form of the Green's function. The farfield $\mathbf{E}_\infty$ can then be expressed in spherical vector components as

$$\mathbf{E}_\infty = \begin{bmatrix} E_\theta \\ E_\phi \end{bmatrix} \;=\; \frac{k^2}{4\pi\varepsilon_o\varepsilon}\,\frac{\exp(ikr)}{r}\,\begin{bmatrix} [p_x\cos\phi + p_y\sin\phi]\cos\theta \;-\; p_z\sin\theta \\ -[p_x\sin\phi - p_y\cos\phi] \end{bmatrix} \tag{12}$$

which renders the familiar farfields of a dipole.