1)

Let

$$\vec{Q}(t) = \vec{A}\cos\omega t + \vec{B}\sin\omega t$$

without loss of generality let

$$\vec{A} = (A_x\hat{x} + A_y\hat{y})$$

$$\vec{B} = (B_x\hat{x} + B_y\hat{y})$$

$$\vec{Q}(t) = (A_x\cos\omega t + B_x\sin\omega t)\,\hat{x} + (A_y\cos\omega t + B_y\sin\omega t)\,\hat{y}$$

$$\vec{Q}(t) = C_0\cos[\omega t+\phi_0]\,\hat{x} + C_1\cos[\omega t+\phi_1]\,\hat{y}$$

Let $\phi_1 = \phi_0 + \phi_a$

$$C_1 = C_0 C_a$$

$$x = C_0\cos[\omega t+\phi_0]$$

then

$$Q(t) = x(t)\,\hat{x} + y(t)\,\hat{y}$$

$$y(t) = C_a\left[C_0\cos[\omega t+\phi_0]\cos\phi_a - C_0\sin[\omega t+\phi_0]\sin\phi_a\right]$$

$$y(t) = C_a\cos\phi_a\,x - C_a\sqrt{C_0^2 - x^2}\,\sin\phi_a$$

$$y - C_a\cos\phi_a\,x = -C_a\sqrt{C_0^2 - x^2}\,\sin\phi_a$$

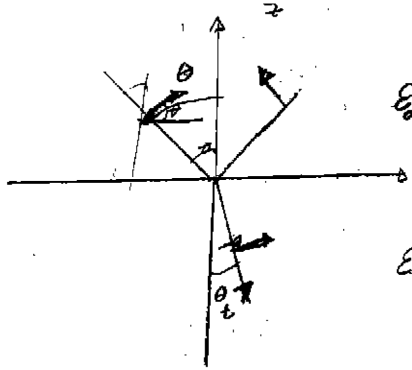
$$y^2 + C_a^2\cos^2\phi_a\,x^2 - 2xy\,C_a\cos\phi_a = C_a^2(C_0^2 - x^2)\sin^2\phi_a$$

$$y^2 + C_a^2x^2 - 2xy\,C_a\cos\phi_a = C_a^2\sin^2\phi_a\,C_0^2$$

$$\boxed{\frac{y^2}{C_a^2} + x^2 - 2xy\,C_a\cos\phi_a = \sin^2\phi_a\,C_0^2}$$

Ellipse equation

2



$\varepsilon_a = \varepsilon = 2.25 = n^2$

$\varepsilon_0 = 1$

$\bar{E}_0 = E_{ox}\cos\theta\,\hat{x} + E_{oy}\sin\theta\,\hat{y}$ $\qquad E_0 = \sqrt{E_{ox}^2 + E_{oy}^2}$

$$E_t = \left(t^p E_0 \frac{k_{x_t}}{k_t}\hat{x} + t E_0 \frac{k_{x\theta}}{k_t}\hat{y} \right) e^{i(k_x x + k_z z)}$$

$$k_{xt} = k_0 \sin\theta_t = k_0\sqrt{\varepsilon}\sin\theta$$

$$k_{zt} \quad k_0\cos\theta_t = k_0\sqrt{1-\sin^2\theta_t} = k_0\sqrt{1-\varepsilon\sin^2\theta}$$

$$t^p = \frac{2 k_{z_1}\sqrt{\varepsilon}}{k_{z_1} + \varepsilon k_{z_t}}$$

then

$$w_e = \frac{1}{2}\left|E_t(\theta)\right|^2$$

$$w_e = \frac{\varepsilon_0 |E_0|^2}{2}\left[\frac{|t_p|^2 |k_{xt}|^2}{k_0^2} + \frac{|t_p|^2 |k_{z}|^2}{k_0^2}\right]\left|e^{-i(k_x x + k_z z)}\right|^2$$

the critical angle is:

$$\sqrt{\varepsilon}\sin\theta_c = 1 \qquad \sin\theta_c = \frac{1}{\sqrt{\varepsilon}} \qquad k_0 = \frac{2\pi}{\lambda_0}$$

$$t^p = \frac{2n\,n\sqrt{1-\sin^2\theta}}{n\sqrt{1-\sin^2\theta} + n^2\sqrt{1-\sin^2\theta_t}} = \frac{2n^2\cos\theta}{n\cos\theta + n^2\sqrt{1-n^2\sin^2\theta}}$$

$$t^p = \frac{2n\cos\theta}{\cos\theta + n\sqrt{1-n^2\sin^2\theta}}$$

then

for $\theta < \theta_c$

$$w_e(\theta) = \frac{E_0^2}{2}|t^p|^2 = \frac{E_0^2\, 4n^2\cos^2\theta}{2\left[\cos\theta + n\sqrt{1-n^2\sin^2\theta}\right]^2}$$

For $\theta > \theta_c$

$$t^p = \frac{2n\cos\theta}{\cos\theta + i n\sqrt{n^2\sin^2\theta - 1}}$$

$k_{x_t} = k_0 \sqrt{\varepsilon} \sin\theta$

$k_z = i k_0 \sqrt{\varepsilon \sin^2\theta - 1}$

$$w_e = \frac{\varepsilon_0 E_0^2 |t_p|^2}{2} \left[\varepsilon \sin^2\theta + \varepsilon \sin^2\theta - 1 \right] e^{-2|K||z|}$$

where $K = k_0 \sqrt{\varepsilon \sin^2\theta - 1}$

$$w_e = \frac{E_0^2 |t_p|^2}{2} \left[2n^2 \sin^2\theta - 1 \right] e^{-2K|z|}$$

Therefore

$$w_e = \frac{\varepsilon_0 E_0^2 |t_p|^2}{2} \frac{4n^2 \cos^2\theta \left[2n^2 \sin^2\theta - 1 \right]}{\cos^2\theta + n^4 \sin^2\theta - n^2} e^{-2k_0\sqrt{\varepsilon \sin^2\theta - 1}|z|}$$

thus:

$$w_e(\theta) = \begin{cases} \dfrac{\varepsilon_0 E_0^2}{2} \cdot \dfrac{4n^2\cos^2\theta}{\left[\cos\theta + n\sqrt{1 - n^2\sin^2\theta}\right]^2} & \theta < \theta_c \\[2ex] \dfrac{\varepsilon_0 E_0^2}{2} \dfrac{4n^2\cos^2\theta\left[2n^2\sin^2\theta - 1\right]}{\cos^2\theta + n^4\sin^2\theta - n^2} e^{-2k_0\sqrt{\varepsilon\sin^2\theta - 1}|z|} & \theta > \theta_c \end{cases}$$

p-polarization

For s-polarization

$$w_e = \frac{|E_0|^2}{2}\left[\frac{|t_s|^2|k_{xt}|^2}{k_0^2} + \frac{|t_s|^2|k_{zt}|^2}{k_0^2}\right]\left|e^{i k_{x_t} x + k_{z_t} z}\right|^2$$

$$t^s = \frac{2k_{z_1}}{k_{z_1} + k_{z_2}}$$

Then for $\theta < \theta_c$

$$t^s = \frac{2k_0 n\sqrt{1-\sin^2\theta}}{k_0 n\sqrt{1-\sin^2\theta} + k_0\sqrt{1-n^2\sin^2\theta}}$$

$$t_s = \frac{2n\cos\theta}{n\cos\theta + \sqrt{1-n^2\sin^2\theta}}$$

then

$$w_e = \frac{|E_0|^2}{2}|t_s|^2$$

$$w_e = \frac{|E_0|^2}{2}\frac{4n^2\cos^2\theta}{\left[n\cos\theta + \sqrt{1-n^2\sin^2\theta}\right]^2}$$

For $\theta > \theta_c$

$$t_s = \frac{2n\cos\theta}{n\cos\theta + i\sqrt{n^2\sin^2\theta - 1}}$$

then

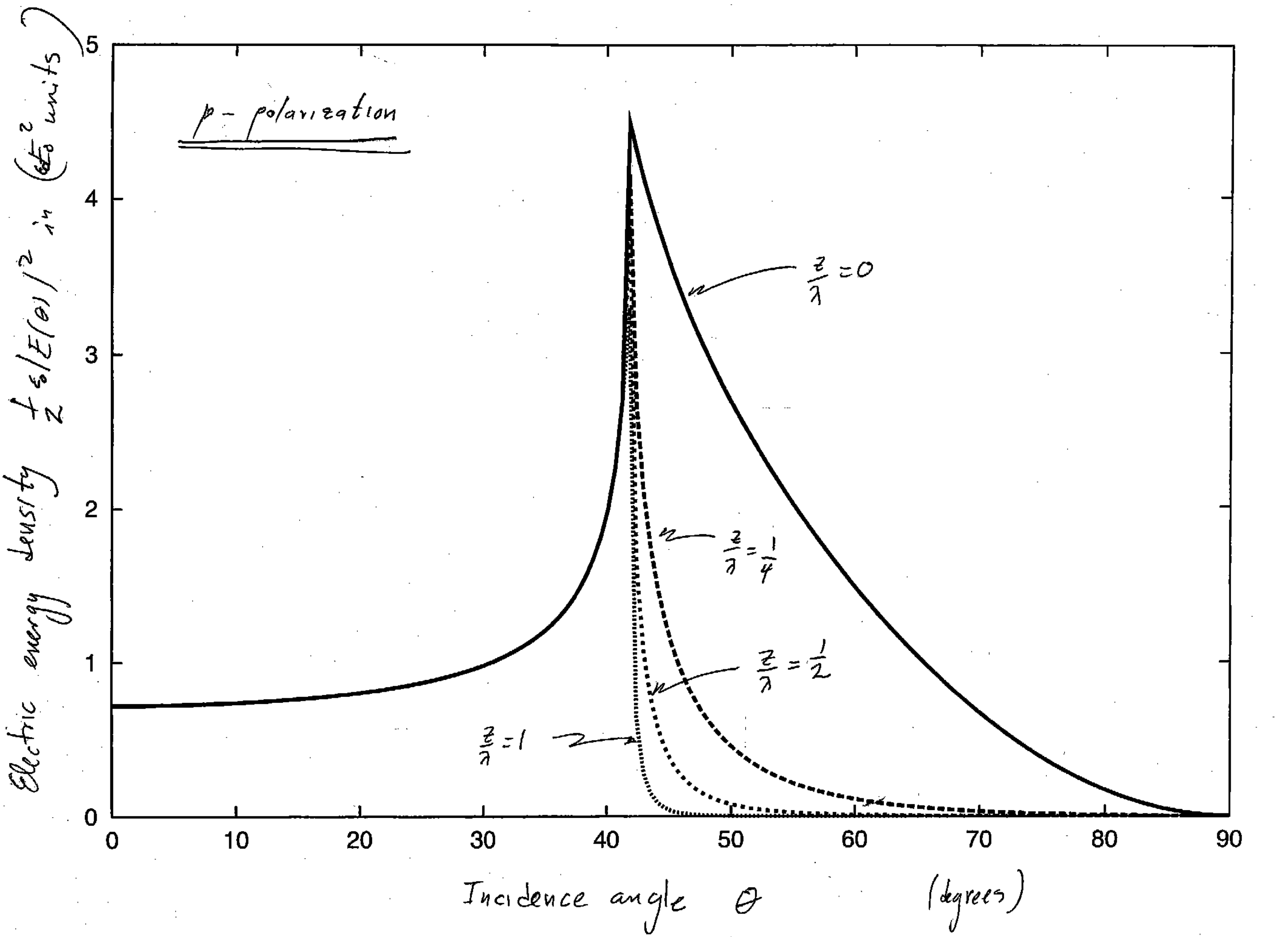
$$w_e = \frac{|E_0|^2}{2}|t_s|^2\left[n^2\sin^2\theta + n^2\sin^2\theta - 1\right]$$

$$w_e = \frac{|E_0|^2}{2}\frac{4n^2\cos^2\theta\left[2n^2\sin^2\theta - 1\right]}{n^2\cos^2\theta + n^2\sin^2\theta - 1}$$

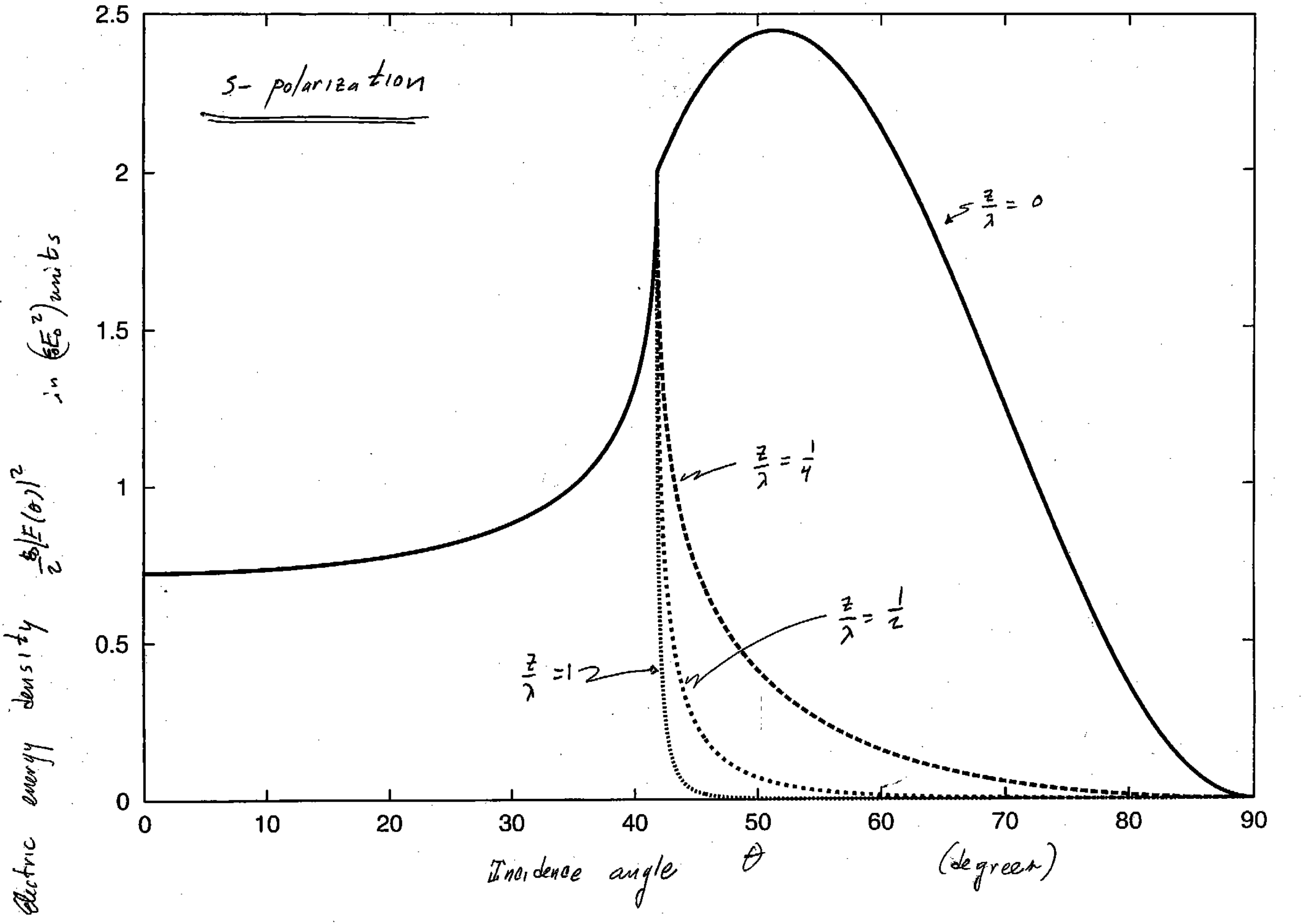
therefore

$$\boxed{w_e(\theta) = \begin{cases} \dfrac{\epsilon_0|E_0|^2}{2}\dfrac{4n^2\cos^2\theta}{\left[n\cos\theta + \sqrt{1-n^2\sin^2\theta}\right]^2} & \text{if } \theta < \theta_c \\[2ex] \dfrac{\epsilon_0|E_0|^2}{2}\dfrac{4n^2\cos^2\theta\left[2n^2\sin^2\theta - 1\right]e^{-k_0\sqrt{\epsilon-\sin^2\theta}\,|z|}}{n^2 - 1} & \text{if } \theta > \theta_c \end{cases}}$$

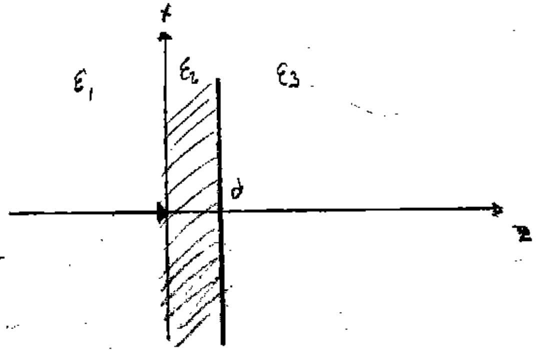
s - polarization



p- polarization
Electric energy density $\frac{1}{2}\varepsilon_0|E(\theta)|^2$ in ($\frac{1}{2}\varepsilon_0 E_0^2$ units)
5
4
3
2
1
0
0
10
20
30
40
50
60
70
80
90
Incidence angle θ (degrees)
$\frac{z}{\lambda}=0$
$\frac{z}{\lambda}=\frac{1}{4}$
$\frac{z}{\lambda}=\frac{1}{2}$
$\frac{z}{\lambda}=1$


s- polarization
$\frac{z}{\lambda} = 0$
$\frac{z}{\lambda} = \frac{1}{4}$
$\frac{z}{\lambda} = \frac{1}{2}$
$\frac{z}{\lambda} = 1$
Electric energy density $\frac{\epsilon_0}{2}|E(\theta)|^2$ in $(\frac{\epsilon_0}{2}E_0^2)$ units
Incidence angle θ (degrees)
0
10
20
30
40
50
60
70
80
90
0
0.5
1
1.5
2
2.5

3



p-polarization

$$\vec{H}_I = \left(A e^{ik_1 z} + B e^{-ik_1 z}\right)\hat{y}\, e^{ik_x x}$$

$$\vec{H}_{II} = \left(C e^{ik_2 z} + D e^{-ik_2 z}\right)\hat{y}\, e^{ik_x x}$$

$$\vec{H}_{III} = F e^{ik_2(z-d)}\hat{y}$$

Boundary conditions

$$\begin{cases} A + B = C + D \\ \dfrac{k_1}{\varepsilon_1}\left[A - B\right] = \dfrac{k_2}{\varepsilon_2}\left[C - D\right] \end{cases}
\qquad
\begin{cases} C e^{ik_2 d} + D e^{-ik_2 d} = F \\ \dfrac{k_2}{\varepsilon_2}\left[C e^{ik_2 d} - D e^{-ik_2 d}\right] = \dfrac{k_3}{\varepsilon_3} F \end{cases}$$

where $K_i = \sqrt{k_0^2 \varepsilon_i - k_x^2}$ $\quad i = 1,2,3$

$k_x = k_0 \sqrt{\varepsilon_1} \sin\theta$

$k_0 = \dfrac{2\pi}{\lambda_0} = \dfrac{\omega}{c}$

then

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \rho_1 & -\rho_1 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix}$$

$$\begin{bmatrix} e^{ik_2 d} & e^{-ik_2 d} \\ e^{ik_2 d} & -e^{-ik_2 d} \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ \rho_2 \end{bmatrix} F$$

$$\boxed{\rho_1 \equiv \frac{K_2 \varepsilon_1}{K_1 \varepsilon_2} \qquad \rho_2 \equiv \frac{K_3 \varepsilon_2}{K_2 \varepsilon_3}}$$

let define

$z_1 \equiv e^{ik_2 d}$ and $z_2 \equiv e^{-ik_2 d}$

$$\begin{bmatrix} 1 \\ \rho_2 \end{bmatrix} F = \begin{bmatrix} z_1 & z_2 \\ z_1 & -z_2 \end{bmatrix}\left\{\begin{bmatrix} 1 & 1 \\ \rho_1 & -\rho_1 \end{bmatrix}\right\}^{-1}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \rho_2 \end{bmatrix} F = \frac{1}{2}\begin{bmatrix} z_1 & z_2 \\ z_1 & -z_2 \end{bmatrix}\begin{bmatrix} 1 & 1/\rho_1 \\ 1 & -1/\rho_1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \rho_2 \end{bmatrix} F = \frac{1}{2}\begin{bmatrix} z_1 & z_2 \\ z_1 & -z_2 \end{bmatrix}\begin{bmatrix} 1 + 1/\rho_1 & 1 - 1/\rho_1 \\ 1 - 1/\rho_1 & 1 + 1/\rho_1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \rho_2 \end{bmatrix} F = \frac{1}{2}\begin{bmatrix} z_1(1 + 1/\rho_1) + z_2(1 - 1/\rho_1) & z_1(1 - 1/\rho_1) + z_2(1 + 1/\rho_1) \\ z_1(1 + 1/\rho_1) - z_2(1 - 1/\rho_1) & z_1(1 - 1/\rho_1) - z_2(1 + 1/\rho_1) \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ \rho_2 \end{bmatrix} F = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}$$

$$\alpha = \frac{1}{2}\left[(z_1+z_2) + \frac{1}{\rho_1}(z_1-z_2)\right]$$

$$\alpha = \cos k_2 d + \frac{i}{\rho_1}\sin k_2 d$$

$$\beta = \frac{1}{2}\left\{[z_1+z_2] - \frac{1}{\rho_1}(z_1-z_2)\right\}$$

$$\beta = \cos k_2 d - \frac{i}{\rho_1}\sin k_2 d$$

$$\gamma = \frac{1}{2}\left[(z_1-z_2) + \frac{1}{\rho_1}(z_1+z_2)\right] =$$

$$\gamma = \frac{1}{\rho_1}\cos k_2 d + i\sin k_2 d$$

$$\delta = \frac{1}{2}\left[(z_1-z_2) - \frac{1}{\rho_1}(z_1+z_2)\right]$$

$$\delta = -\frac{1}{\rho_1}\cos k_2 d + i\sin k_2 d$$

so

$$F = \alpha A + \beta B$$

$$\rho_2 F = \gamma A + \delta B$$

Solving for B in terms of A

$$\rho_2\alpha A + \rho_2\beta B = \gamma A + \delta B$$

$$(\rho_2\alpha - \gamma)A = B(\delta - \beta\rho_2)$$

$$\frac{B}{A} = \frac{\rho_2\alpha - \gamma}{\delta - \beta\rho_2}$$

so

$$\frac{B}{A} = \frac{\rho_2\left[\cos k_2 d + \frac{i}{\rho_1}\sin k_2 d\right] - \left[\frac{1}{\rho_1}\cos k_2 d + i\sin k_2 d\right]}{-\frac{\cos k_2 d}{\rho_1} + i\sin k_2 d - \rho_2\left[\cos k_2 d - \frac{i}{\rho_1}\sin k_2 d\right]}$$

$$\boxed{\frac{B}{A} = \frac{\cos k_2 d\left[\rho_2 - \frac{1}{\rho_1}\right] + i\sin k_2 d\left[\frac{\rho_2}{\rho_1} - 1\right]}{-\cos k_2 d\left[\frac{1}{\rho_1} + \rho_2\right] + i\sin k_2 d\left[1 + \frac{\rho_2}{\rho_1}\right]} = r}$$

Intensity

$$|\langle\vec{S}\rangle| = \left|\frac{1}{2}\mathrm{Re}\{\vec{E}_r \times \vec{H}_r^*\}\right|$$

$$\vec{E}_r = \frac{i}{\omega\epsilon_0\epsilon_1}\nabla\times\vec{H}_r = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = -\hat{x}\frac{\partial H_{y_r}}{\partial z} + \hat{z}\frac{\partial H_{y_r}}{\partial x}$$

$$\vec{E} = \frac{i}{\omega\epsilon_0\epsilon_1}\left\{-ik_{z_1}H_{y_r}\hat{x} + ik_{x_1}H_{y_r}\hat{z}\right\}$$

$$H_{y_r} = H_0 r e^{-ik_{z_1}z}e^{ik_x x}$$

$$\vec{E} = \frac{1}{\omega\epsilon_0\epsilon_1}\left\{k_{z_1}H_{y_r}\hat{x} - k_{x_1}H_{y_r}\hat{z}\right\}$$

$$\langle S\rangle = \frac{1}{2\omega\epsilon_0\epsilon_1}\mathrm{Re}\left\{k_{z_1}H_{y_r}H_{y_r}^*\hat{z} + k_x H_{y_r}H_{y_r}^*\hat{x}\right\}$$

$$|\langle \bar{s}\rangle| = \frac{1}{2\omega\varepsilon_0\varepsilon_1}|H_0|^2|t|^2 k_0\sqrt{\varepsilon_1}$$

$$|\langle s\rangle| = \frac{1}{2\varepsilon_0\sqrt{\varepsilon_1}\,c}|H_d|^2|r|^2 = \frac{1}{2\varepsilon_0\sqrt{\varepsilon_1}\,\mu_0^2}\frac{|E_0|^2\varepsilon_1}{c^3}|r|^2$$

$$|\langle s\rangle| = \frac{1}{2}\varepsilon_0 c\sqrt{\varepsilon_1}|E_0|^2|r|^2 = \frac{1}{2}\varepsilon_0\varepsilon_1\frac{c}{\sqrt{\varepsilon_1}}|E_0|^2|r|^2$$

$$\boxed{|\langle s\rangle| = \frac{1}{2}\varepsilon_0\varepsilon_1\frac{c}{\sqrt{\varepsilon_1}}|E_0|^2|r|^2}$$

It was a problem about 'surface plasmons ~~in one~~ of the first homework set. In that problem we concluded that in a planar interface surface plasmons can "not" be produced by radition modes. [redacted] is only evanescent modes can couple to the plasmon modes. The present configuration allows to generated plasmons with optical waves.

The plasmons are determined by finding the poles of r, namely

$$-\cos k_{z_p}d\left[\frac{1}{\rho_1}+\rho_2\right] + i\sin k_{z_p}d\left[1+\frac{\rho_2}{\rho_1}\right] = 0$$

where

$$k_{z_p} = \sqrt{\varepsilon_2\frac{\omega^2}{c^2} - k_{x_p}^2}$$

In general the solution ~~is a~~ complex number k_{xp}

$$k_{xp} = k_{x_{r_p}} + i\,k_{x_{i_p}}$$

Obviosly the location of the poles depend on the distance $\underline{\underline{d}}$ and frequency $\underline{\underline{\omega}}$.

The idea is to look for material such that $\underline{k_{x_{i_p}}}$ be $\underline{\text{small}}$ and that

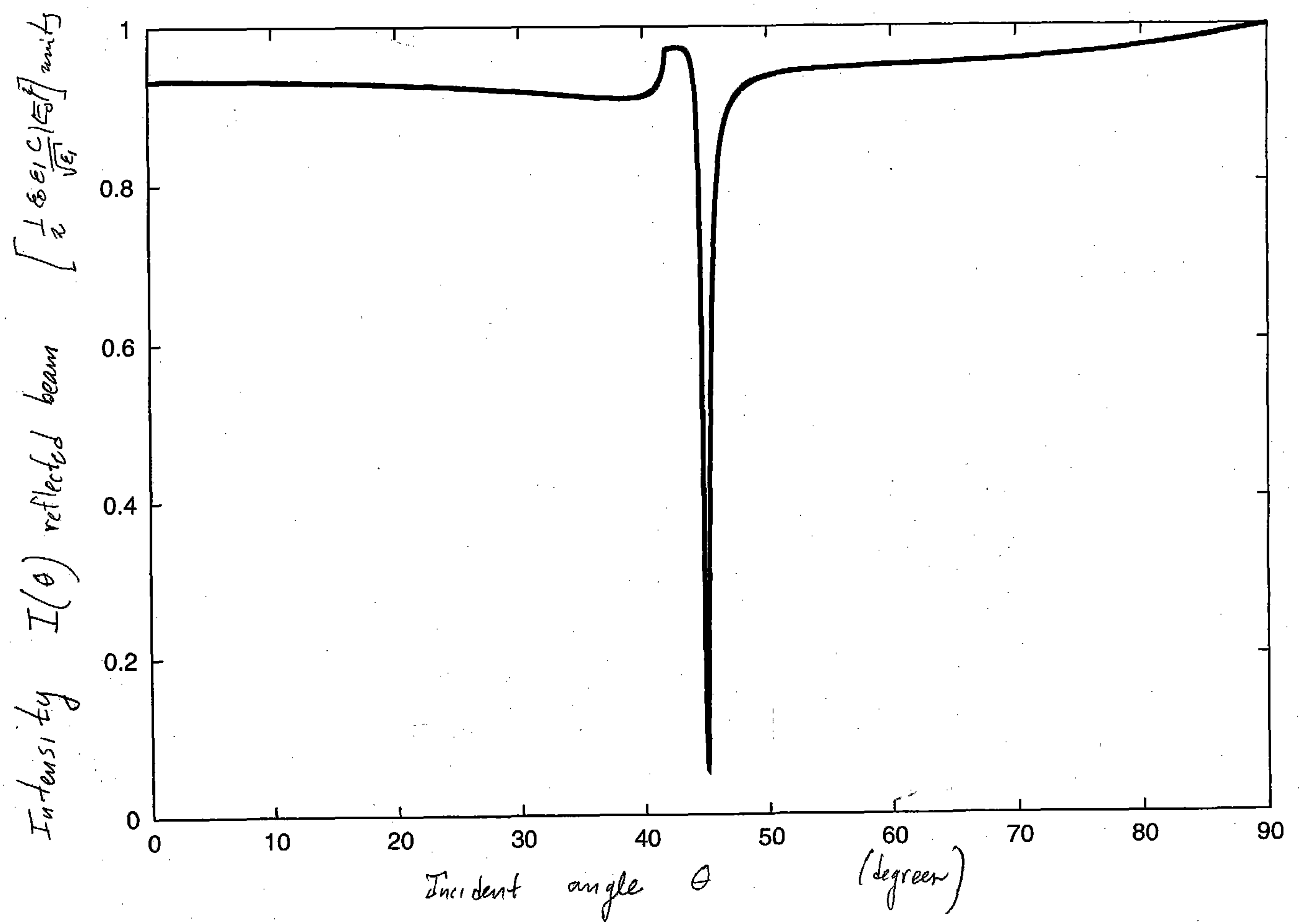
$$k_{x\,r_p} \approx \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_1}\sin\theta$$

In this example the resonance condition is matched nearly for

$$\theta_k = 0.785 \text{ rad} \quad \text{or} \quad 45.02^\circ$$

Thus the energy is coupled to the plasmon.

On the other hand a plasmon can only radiated in the far zone only this angle θ_k


Intensity I(θ) reflected beam [½ ε0 ε1 c/√ε1 |E0|²] unit
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
Incident angle θ (degree)