Homework 1

1

Gauss' Theorem

$$\oiint_{\partial V} \vec{F}\cdot d\vec{a} = \iiint_V \nabla\cdot\vec{F}\, dV \qquad [1]$$

Stokes' Theorem

$$\oint_{\partial S} \vec{F}\cdot d\vec{l} = \iint_S \nabla\times\vec{F}\cdot d\vec{a} \qquad [2]$$

Here $\vec{F}$ is a vectorial field.

Differential form of Maxwell's equations

$$\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad [3]$$

$$\nabla\times\vec{B} = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad [4]$$

$$\nabla\cdot E = \rho/\epsilon_0 \qquad [5]$$

$$\nabla\cdot B = 0 \qquad [6]$$

<u>Intergral Forms</u>

For eq [3].

$$\iint_S (\nabla\times E)\cdot d\vec{a} = \oint_{\partial S} \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot d\vec{a} \qquad [7]$$

$$\therefore \quad \oint_{\partial S} \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\Phi_m \qquad [8]$$

where Φ_m is the magnetic Flux

$$\Phi_m = -\iint_S \vec{B}\cdot d\vec{a} \qquad [9]$$

For eq [4]

$$\iint_S \nabla\times\vec{B}\cdot d\vec{a} = \oint_{\partial S}\vec{B}\cdot d\vec{l} = \mu_0\iint_S \vec{J}\cdot d\vec{a} + \mu_0\epsilon_0\frac{\partial}{\partial t}\iint \vec{E}\cdot d\vec{a} \qquad [10]$$

$$\oint_{\partial S}\vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\epsilon_0\frac{\partial}{\partial t}\Phi_e \qquad [11]$$

Here I is the current through the surface S and Φ_e is the electric flux through surface S.

For eq [5]

$$\iiint_V \nabla\cdot\vec{E}\, dV = \oiint_{\partial V}\vec{E}\cdot d\vec{a} = \iiint_V \frac{\rho}{\epsilon_0}\, dV \qquad [12]$$

$$\oiint_{\partial V}\vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0} \qquad [13]$$

where Q is the charge contained in V

For eq [6]

$$\iiint_V \nabla\cdot\vec{B}\, dV = \oiint_{\partial V}\vec{B}\cdot d\vec{a} = 0 \qquad [14]$$

$$\oiint_{\partial V}\vec{B}\cdot d\vec{a} = 0 \qquad [15]$$

Comments

When using the quantities $\vec{D}$ and $\vec{H}$, the quantities ρ_f and $\vec{J}_f$ are the free charge and free density current respectively.

$$\nabla\cdot\vec{E} = \frac{\rho_f}{\epsilon_0} + \frac{\rho_{pol}}{\epsilon_0} = \frac{\rho_f}{\epsilon_0} - \frac{\nabla\cdot\vec{P}}{\epsilon_0}$$

$$\nabla\cdot\epsilon_0\vec{E} = \rho_f - \nabla\cdot\vec{P}$$

$$\nabla\cdot\{\epsilon_0\vec{E} + \vec{P}\} = \rho_f$$

$$\nabla\cdot\vec{D} = \rho_f$$

$$\nabla\times\frac{\vec{B}}{\mu_0} = \vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\frac{\vec{B}}{\mu_0} = \underbrace{\vec{J}_f}_{\text{free}} + \underbrace{\vec{J}_M}_{\text{magnetisation}} + \underbrace{J_{Pol}}_{\text{polarization}} + \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\frac{\vec{B}}{\mu_0} = \vec{J}_f + \nabla\times\vec{M} + \frac{\partial\vec{P}}{\partial t} + \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\left[\frac{\vec{B}}{\mu_0} - \vec{M}\right] = \vec{J}_f + \frac{\partial}{\partial t}\{\epsilon_0\vec{E} + \vec{P}\}$$

$$\nabla\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t}$$

2/ [#1]

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad [1]$$

$$\nabla \cdot E = 0 \qquad [2]$$

$$\nabla \cdot B = 0 \qquad [3]$$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad [4]$$

Taking the curl of eq [1]

$$\nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E} + \nabla(\nabla \cdot \vec{E}) = -\frac{\partial}{\partial t} \nabla \times B \qquad [5]$$

using eqs [2] and [4] into eq [5]

$$-\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad [6]$$

If $\vec{E} = E_x \hat{i} + 0\hat{j} + 0\hat{z}$, the eq [6] becomes

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \qquad [7]$$

and $\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} = \frac{\partial f(z-ct)}{\partial x} = 0$ } fulfills eq. [2]

Let define λ as

$$\lambda \equiv z - ct \qquad [8]$$

if $E_x = f(\lambda)$ satisfy maxwell's equation then E_x must fulfill eq [7]

$$\frac{\partial E_x}{\partial z} = \frac{df(\lambda)}{d\lambda}\frac{\partial \lambda}{\partial z} = \frac{df(\lambda)}{d\lambda} \qquad [9]$$

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial E_x}{\partial z}\right] = \frac{d}{d\lambda}\left[\frac{\partial f(\lambda)}{d\lambda}\right]\frac{\partial \lambda}{\partial z} \qquad [10]$$

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{d^2 f(\lambda)}{d\lambda^2} \qquad [11]$$

In the other hand

$$\frac{\partial E_x}{\partial t} = \frac{df(\lambda)}{d\lambda}\frac{\partial \lambda}{\partial t} = -c\frac{df(\lambda)}{d\lambda} \qquad [12]$$

$$\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial E}{\partial t}\right] = \frac{d}{d\lambda}\left[\frac{\partial E_x}{\partial t}\right]\frac{\partial \lambda}{\partial t} = \frac{d}{d\lambda}\left[-c\frac{df(\lambda)}{d\lambda}\right](-c) = c^2\frac{d^2 f}{d\lambda^2}$$

$$\frac{\partial^2 E_x}{\partial t^2} = c^2\frac{d^2 f(\lambda)}{d\lambda^2} \qquad [14]$$

From eqs [14] and [11] comparing with eq [7] we conclude that

$f(z-ct)$ is a solution of eq [7] if

$$c^2 \equiv \frac{1}{\mu_0 \epsilon_0} \qquad [15]$$

Therefore

$E_x = f(z-ct)$ fulfill maxwell's equation without sources and propagating in vacuum

$$\text{where} \quad c^2 \equiv \frac{1}{\mu_0 \epsilon_0} \qquad [16]$$

-2] From eq [1]

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \hat{y}\frac{\partial E_x}{\partial z}$$

thus

$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} \quad \Rightarrow B_y \text{ depends on } z$$

From Maxwell's equation using a similar procedure as for the electric field, we obtain

$$\frac{\partial^2 B_y}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_y}{\partial t^2}$$

Then $B_y = g(z-ct)$ is a solution

where $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

Therefore from eq

$$\frac{\partial B_y}{\partial t} = \frac{dg(s)}{ds}\frac{\partial s}{\partial t} = -\frac{dg(s)}{ds}c$$

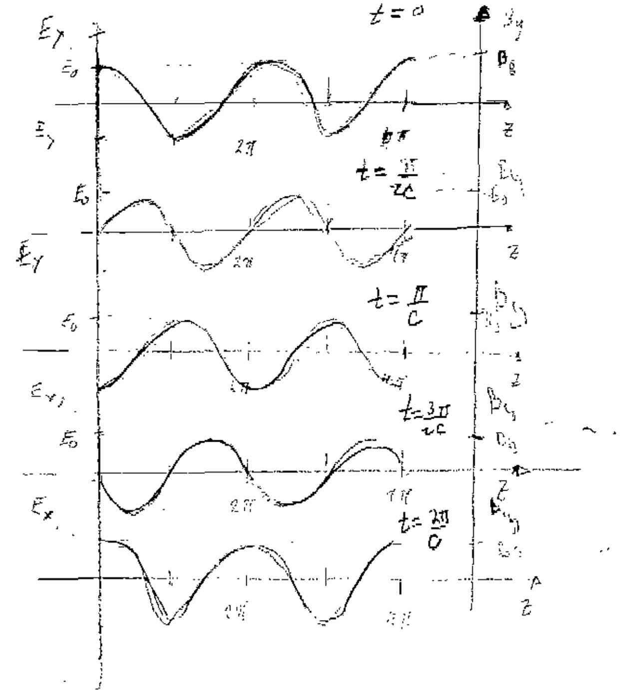
$$\frac{\partial B_y}{\partial t} = -\frac{dg(s)}{ds}c = -\frac{\partial E_x}{\partial z} = -\frac{df(s)}{ds}$$

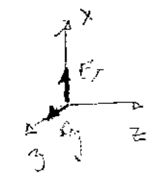
$$\Rightarrow g(s) = \frac{1}{c}f(s)$$

$\therefore$

$$B_y = \frac{1}{c}f(z-ct)$$

-3]





$$E_x = \cos[z-ct]$$
$$B_y = \frac{1}{c}\cos[z-ct]$$

3

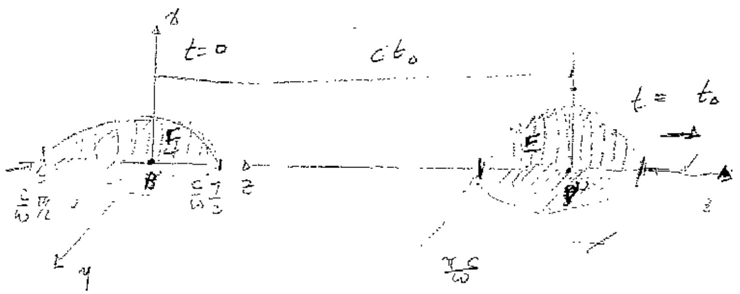
3.1 - From the expressions that were derived in problem 2.1 and 2.2

$$B_y = \frac{1}{c} E_0 f\left[\frac{\omega}{c}(z-ct)\right]$$

then

$$B_y = \begin{cases} \frac{E_0}{c}\cos^2\left[\frac{\omega}{c}(z-ct)\right] & -\frac{\pi}{2} < \frac{\omega}{c}(z-ct) < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases}$$

3.2



The pulse does not change the shape as it propagates with a velocity c. Therefore the region in which the fields are not zero is still $-\pi/2 < \frac{\omega}{c}(z-ct) < \pi/2$

3.3

The Poynting vector $\vec{S}$

$$\vec{S} = \vec{E} \times \vec{H} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{E_x B_y}{\mu_0}\hat{z}$$

$$\vec{S} = \begin{cases} \frac{E_0^2}{c\mu_0}\cos^4\left[\frac{\omega}{c}(z-ct)\right]\hat{z} & -\frac{\pi}{2} < \frac{\omega}{c}(z-ct) < \pi/2 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore the energy is propagating in the positive $\hat{z}$ direction.