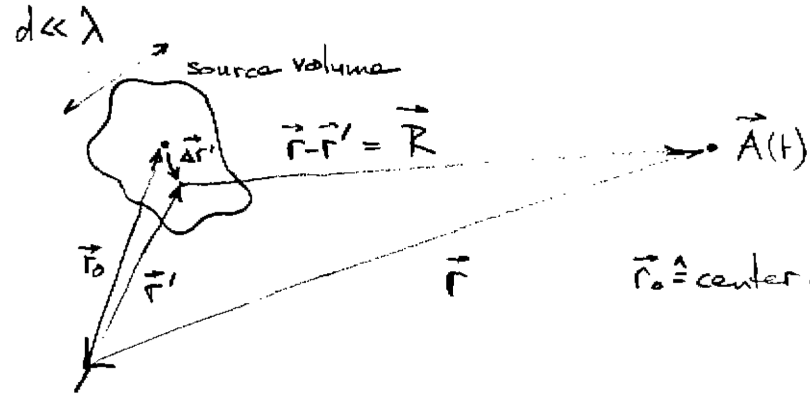
MULTIPOLE EXPANSION OF RADIATING SOURCES



$\vec{r}_0 \hat{=}$ center off current distribution

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \int_t \int_V \frac{\vec{j}(\vec{r}\,',t')}{|\vec{r}-\vec{r}\,'|}\, \delta\left[t' - \left(t - R/c\right)\right] dV'\, dt' \qquad (1)$$

1. time-harmonic current distribution: $\vec{j}(\vec{r}\,',t') = \mathrm{Re}\left\{\vec{j}(\vec{r}\,')\, e^{-i\omega t'}\right\}$

$$\longrightarrow \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{j}(\vec{r}\,')\, \frac{e^{ikR}}{R}\, dV' \qquad \left(\vec{r}\,' = \vec{r}_0 + \Delta\vec{r}\,'\right) \qquad (2)$$

2. $d \ll R$:
$$R = |\vec{r}-\vec{r}\,'| = \left|(\vec{r}-\vec{r}_0) - \Delta\vec{r}\,'\right| = |\vec{r}-\vec{r}_0| \left[1 - \frac{2(\vec{r}-\vec{r}_0)\cdot\Delta\vec{r}\,'}{|\vec{r}-\vec{r}_0|^2} + \frac{\Delta r'^2}{|\vec{r}-\vec{r}_0|^2}\right]^{1/2}$$

$$\approx |\vec{r}-\vec{r}_0| \left[1 - 2\,\frac{(\vec{r}-\vec{r}_0)\cdot\Delta\vec{r}\,'}{|\vec{r}-\vec{r}_0|^2}\right]^{1/2}$$

$$\approx |\vec{r}-\vec{r}_0| \left[1 - \frac{(\vec{r}-\vec{r}_0)\cdot\Delta\vec{r}\,'}{|\vec{r}-\vec{r}_0|^2}\right] \qquad (3)$$

$$\rightarrow \vec{A}(\vec{r}) = \mu_0 \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|} \int_V \vec{j}(\vec{r}_0+\vec{\Delta r}') \, e^{-ik\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|}\cdot\vec{\Delta r}'} \, d^3\vec{\Delta r}' \qquad (4)$$

(3) $d \ll \lambda$: $$e^{-ik\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|}\cdot\vec{\Delta r}'} = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \left[\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|}\cdot\vec{\Delta r}'\right]^n$$

$$\boxed{\vec{A}(\vec{r}) = \mu_0 \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|} \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \int_V \vec{j}(\vec{r}_0+\vec{\Delta r}') \left[\frac{(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|}\cdot\vec{\Delta r}'\right]^n d^3\vec{\Delta r}'} \qquad (5)$$

(4) lowest order term: $n=0$ $\rightarrow$ dipole radiation

use: $$\int_V \vec{j}(\vec{r}_0+\vec{\Delta r}')\, d^3\vec{\Delta r}' = \int_V (\vec{r}_0+\vec{\Delta r}') \underbrace{\left[\nabla\cdot\vec{j}(\vec{r}_0+\vec{\Delta r}')\right]}_{i\omega\rho(\vec{r}_0+\vec{\Delta r}')} d^3\vec{\Delta r}'$$

$$\boxed{\vec{A}(\vec{r}) = -i\omega\mu_0 \vec{p}\, \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|} \quad \text{with} \quad \vec{p} = \int_V (\vec{r}_0+\vec{\Delta r}')\, \rho(\vec{r}_0+\vec{\Delta r}')\, d^3\vec{\Delta r}'} \qquad (6)$$

$\vec{p} \;\hat{=}\;$ dipole moment (electrical)

(5) in terms of Green's function: $$\boxed{\vec{A}(\vec{r}) = -i\omega\mu_0 \vec{p}\, G_0(\vec{r},\vec{r}_0)} \qquad (7)$$

thus: $\vec{j}(\vec{r}') = -i\omega\vec{p}\, \delta(\vec{r}'-\vec{r}_0)$

6) Electric field of dipole:

$$\vec{E}(\vec{r}) = i\omega \left[1 + \frac{1}{k^2}\nabla\nabla\cdot\right] \vec{A}(\vec{r}) \quad \text{(Lorentz gauge)} \tag{8}$$

$$= \omega^2\mu_0 \left[1 + \frac{1}{k^2}\nabla\nabla\cdot\right]\left(G_0(\vec{r},\vec{r}_0)\,\vec{p}\right)$$

$$= \omega^2\mu_0 \underbrace{\left\{\left[\overleftrightarrow{I} + \frac{1}{k^2}\nabla\nabla\right] G_0(\vec{r},\vec{r}_0)\right\}}_{=\overleftrightarrow{G}(\vec{r},\vec{r}_0) \text{ (dyadic Green's function)}} \vec{p}$$

thus: $$\boxed{\vec{E}(\vec{r}) = \omega^2\mu_0\, \overleftrightarrow{G}(\vec{r},\vec{r}_0)\,\vec{p}} \tag{9}$$

7) Magnetic field of dipole:

$$\vec{H}(\vec{r}) = \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{r}) \tag{10}$$

$$= -i\omega\, \nabla\times\left(G_0(\vec{r},\vec{r}_0)\,\vec{p}\right)$$

$$= -i\omega\left[\nabla\times\overleftrightarrow{G}(\vec{r},\vec{r}_0)\right]\vec{p}$$

or, similarily: $\vec{H} = \frac{1}{i\omega\mu_0}\nabla\times\vec{E}$

with Eq. (9):

$$\boxed{\vec{H}(\vec{r}) = -i\omega\left[\nabla\times\overleftrightarrow{G}(\vec{r},\vec{r}_0)\right]\vec{p}} \tag{11}$$