THE LIÉNARD - WIECHERT POTENTIALS

Vector- and Scalar potentials :

$$\vec{A}(\vec{r},t) = \mu_0 \int_V \int_t G_0(\vec{r},\vec{r}';t,t')\,\vec{j}(\vec{r}',t')\,dt'\,dV' \quad (1)$$

$$\phi(\vec{r},t) = \frac{1}{\varepsilon_0} \int_V \int_t G_0(\vec{r},\vec{r}';t,t')\,\rho(\vec{r}',t')\,dt'\,dV' \quad (2)$$

Scalar Green's function :

$$G_0(\vec{r},\vec{r}';t,t') = \frac{\delta\left[t' - \left(t - |\vec{r}-\vec{r}'|\frac{1}{c}\right)\right]}{4\pi\,|\vec{r}-\vec{r}'|} \quad (3)$$

→ retarded, free-space, i.e $n=1$

Insert G_0 and integrate over t' :

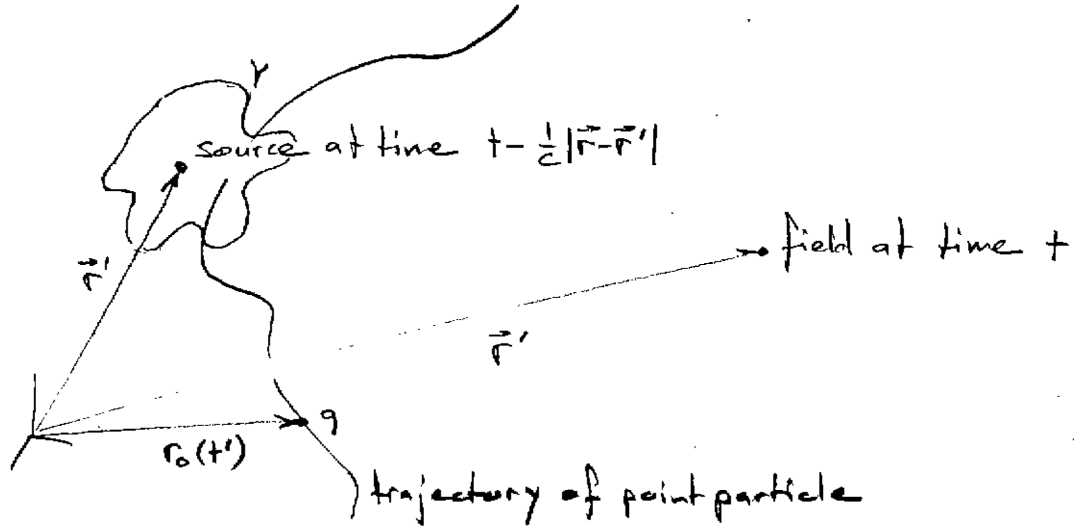
$$\vec{A}(\vec{r},t) = \mu_0 \int_V \frac{\vec{j}(\vec{r}',t'=t-\frac{1}{c}|\vec{r}-\vec{r}'|)}{4\pi\,|\vec{r}-\vec{r}'|}\,dV' \quad (4)$$

$$\phi(\vec{r},t) = \frac{1}{\varepsilon_0} \int_V \frac{\rho(\vec{r}',t'=t-\frac{1}{c}|\vec{r}-\vec{r}'|)}{4\pi\,|\vec{r}-\vec{r}'|}\,dV' \quad (5)$$

Retarded time :

$$t' = t - \frac{1}{c}|\vec{r}-\vec{r}'|$$

The field at $\vec{r}$ at an instant of time t is determined by the charges and currents at $\vec{r}'$ evaluated at the earlier time $t-\frac{1}{c}|\vec{r}-\vec{r}'|$!



Consider point particle with coordinate $\vec r_0$ and velocity $\vec v_0$:

$$\rho(\vec r', t') = q\,\delta[\vec r' - \vec r_0(t')] \tag{6}$$

$$\vec j(\vec r', t') = q\,\vec v_0(t')\,\delta[\vec r' - \vec r_0(t')] \tag{7}$$

Insert in (1) and (2) and use (3) :

$$\vec A(\vec r, t) = q\mu_0 \int_t \frac{\vec v_0(t')\ \delta\left[t'-t+|\vec r-\vec r_0(t')|\frac{1}{c}\right]}{4\pi\,|\vec r-\vec r_0(t')|}\,dt' \tag{8}$$

$$\phi(\vec r, t) = \frac{q}{\varepsilon_0}\int_t \frac{\delta\left[t'-t+|\vec r-\vec r_0(t')|\frac{1}{c}\right]}{4\pi\,|\vec r-\vec r_0(t')|}\,dt' \tag{9}$$

Substitution : $\quad t_0 \equiv t' - t + \frac{1}{c}|\vec r - \vec r_0(t')| \qquad (10)$

$$\rightarrow \frac{dt_0}{dt'} = \underbrace{\frac{dt'}{dt'}}_{=1} - \underbrace{\frac{dt}{dt'}}_{=0} + \frac{1}{c}\frac{d}{dt'}|\vec r - r_0(t')| \qquad (t \text{ fixed})$$

$$= \frac{1}{c}\left\{\frac{\partial}{\partial x_0}|\vec r-\vec r_0|\frac{dx_0}{dt'} + \frac{\partial}{\partial y_0}|\vec r-\vec r_0|\frac{dy_0}{dt'} + \frac{\partial}{\partial z_0}|\vec r-\vec r_0|\frac{dz_0}{dt'}\right\}$$

since $\frac{d\vec r}{dt'} = 0 \qquad (\vec r \text{ fixed})$

$$= \frac{1}{c}\left(\nabla_{\vec r_0}|\vec r-\vec r_0|\right)\cdot\frac{d\vec r_0}{dt'}$$

thus: $$\frac{dt_0}{dt'} = 1 + \frac{1}{c}\left(\nabla_{\vec r_0}|\vec r - \vec r_0|\right)\cdot \underbrace{\frac{d\vec r_0}{dt'}}_{\vec V_0}$$

$$\longrightarrow \nabla_{\vec r_0}|\vec r - \vec r_0| = \frac{\partial}{\partial x_0}\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}\ \vec n_x + \cdots$$

$$= -\frac{(x-x_0)}{|\vec r - \vec r_0|}\vec n_x + \frac{(y-y_0)}{|\vec r - \vec r_0|}\vec n_y + \frac{(z-z_0)}{|\vec r - \vec r_0|}\vec n_z$$

$$= -\frac{\vec r - \vec r_0}{|\vec r - \vec r_0|}$$

$$\longrightarrow dt_0 = dt'\left[1 - \frac{\vec V_0}{c}\cdot\frac{(\vec r - \vec r_0)}{|\vec r - \vec r_0|}\right] \qquad (11)$$

Insert in (8) and (9) and use (10) :

$$\vec A(\vec r, t) = q\mu_0 \int_t \frac{\vec V_0\ \delta(t_0)\,|\vec r - \vec r_0|}{|\vec r - \vec r_0| - \frac{\vec V_0}{c}\cdot(\vec r - \vec r_0)}\ \frac{1}{4\pi|\vec r - \vec r_0|}\,dt_0$$

$$= \frac{q\mu_0}{4\pi}\ \frac{\vec V_0}{|\vec r - \vec r_0| - \frac{\vec V_0}{c}\cdot(\vec r - \vec r_0)}\Bigg|_{t_0=0}$$

$$\longrightarrow t_0 = 0 \Rightarrow t' = t - \frac{1}{c}|\vec r - \vec r_0(t')|$$

similarily for $\phi(\vec r, t)$

Finally :

$$\boxed{\begin{aligned} \vec{A}(\vec{r},t) &= \frac{q\mu_0}{4\pi}\,\frac{\vec{v}_0}{|\vec{r}-\vec{r}_0| - \frac{\vec{v}_0}{c}\cdot(\vec{r}-\vec{r}_0)} \\ \phi(\vec{r},t) &= \frac{q}{4\pi\varepsilon_0}\,\frac{1}{|\vec{r}-\vec{r}_0| - \frac{\vec{v}_0}{c}\cdot(\vec{r}-\vec{r}_0)} \\ &\text{evaluated at } t' = t - \tfrac{1}{c}|\vec{r}-\vec{r}_0(t')| \end{aligned}} \qquad (12)$$

$\vec{r}_0 = \vec{r}_0(t')$

$\vec{v}_0 = \vec{v}_0(t')$

The effect of a moving particle is a modification of the denominator, i.e

$$|\vec{r}-\vec{r}_0| \longrightarrow |\vec{r}-\vec{r}_0| - \frac{\vec{v}_0}{c}\cdot(\vec{r}-\vec{r}_0)$$

$\vec{A}$ and ϕ are increased (decreased) if the charge moves towards (away from) the observation point $\vec{r}$.

Equations are relativistically correct : $\vec{v}_0$ = relative velocity

Calculate electric and magnetic fields :

$$\vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r},t) - \nabla\phi(\vec{r},t)$$

$$\vec{B}(\vec{r},t) = \nabla\times\vec{A}(\vec{r},t)$$

$$\longrightarrow \boxed{\begin{aligned} \vec{E}(\vec{r},t) &= \frac{q}{4\pi\varepsilon_0}\left[\frac{R\,(c\hat{\vec{R}}-\vec{v}_0)(c^2-v_0^2)}{(\vec{R}\cdot[c\hat{\vec{R}}-\vec{v}_0])^3} + \frac{R\,[\vec{R}\times([c\hat{\vec{R}}-\vec{v}_0]\times\vec{a}_0)]}{(\vec{R}\cdot[c\hat{\vec{R}}-\vec{v}_0])^3}\right] \\ &\text{where } \vec{R} = (\vec{r}-\vec{r}_0),\ \vec{a}_0 = \frac{d\vec{v}_0}{dt'} = \frac{d^2\vec{r}\,'}{dt'^2},\ \hat{\vec{R}} = \frac{\vec{R}}{R} \end{aligned}} \quad (13)$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}\left(\frac{\vec{R}}{R} \times \vec{E}(\vec{r},t)\right) \tag{14}$$

First term in (13) is a velocity term that decays as $\frac{1}{R^2}$.

Second term in (13) is an acceleration term that decays as $\frac{1}{R}$.

$\longrightarrow$ Radiation $\left(\frac{1}{R}\right)$ is generated by acceleration of charge!

Radiation field: $$\vec{E}_{rad}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{R}{\left(\vec{R}\cdot[c\hat{\vec{R}} - \vec{v}_0]\right)^3} \vec{R} \times \left([c\hat{\vec{R}} - \vec{v}_0] \times \vec{a}_0\right) \tag{15}$$

$$\vec{E}_{rad} \cdot \vec{R} \propto \underbrace{\left(\vec{R} \times [c\hat{\vec{R}} - \vec{v}_0] \times \vec{a}_0\right)}_{(\vec{R}\cdot\vec{a}_0)[c\hat{\vec{R}} - \vec{v}_0] - (\vec{R}\cdot[c\hat{\vec{R}} - \vec{v}_0])\,\vec{a}_0} \cdot \vec{R}$$

$$\propto (\vec{R}\cdot\vec{a}_0)[cR - \vec{v}_0\cdot\vec{R}] - [cR - \vec{v}_0\cdot\vec{R}](\vec{a}_0\cdot\vec{R}) = 0$$

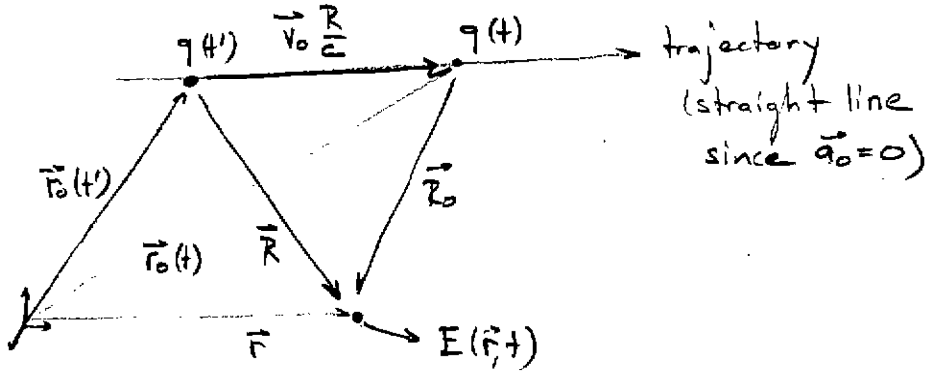
thus: $\boxed{\vec{E}_{rad}\cdot\vec{R} = 0}$ same for $\vec{B}_{rad}$ (16)

The radiation field is <u>transverse</u>!

Special cases:

① $\vec{a}_o = 0, \ \vec{V}_o = 0$:
$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\vec{R}}{R^3} \qquad \text{(Coulomb's law)} \qquad (17)$$

② $\vec{a}_o = 0, \ \vec{V}_o \neq 0$:



$\vec{R} = |\vec{r} - \vec{r}_o(t')|$ retarded distance

$\vec{R}_o = |\vec{r} - \vec{r}_o(t)|$ actual distance

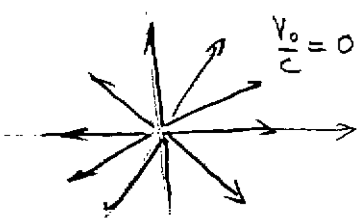
The directed distance $\vec{V}_o \frac{R}{c}$ is the distance that the particle traveled while field propagated from retarded position to observer at $\vec{r}$.

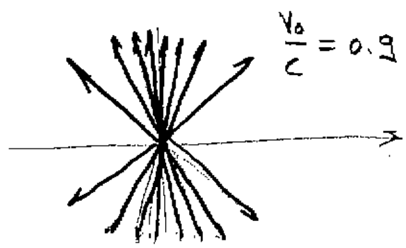
$$[c\hat{\vec{R}} - \vec{V}_o] = \frac{c}{R}[\vec{R} - \vec{V}_o\frac{R}{c}] = \frac{c}{R}\vec{R}_o$$

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{c(c^2 - V_o^2)}{(Rc - \vec{V}_o\cdot\vec{R})^3}\vec{R}_o \qquad (18)$$

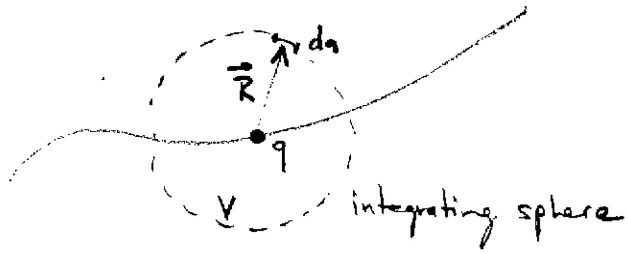
Field is radial (longitudinal) from the charge's <u>actual position</u> (not retarded position).

Field distribution determined by $\vec{V}_o$:





RADIATION FROM MOVING CHARGE



Energy per unit time interval dt crossing surface element da : $\dfrac{dW}{dt}\, da = \vec{S}\cdot\vec{n}\, da$ (19)

careful: Because of movement the rate at which energy leaves the sphere V is <u>not</u> the same as the rate at which it leaves the charge!

The Poynting vector $\vec{S}$ on the surface of the sphere is defined by the fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$. Therefore, Eq. (19) gives the energy flux at time t. This is different from the actual time t' at the position of the charge.

$$\frac{dW}{dt} = \frac{dW}{dt'}\,\frac{dt'}{dt} \tag{20}$$

rate at which energy crosses surface element da

rate at which charge loses energy

$\vec{S}$ is energy per unit area (da) per unit time interval (dt). Not dt'!

Power crossing sphere : $P_s = \int_{\partial V} \frac{dW}{dt}\, da$ (21)

Power radiated by charge : $P = \int_{\partial V} \frac{dW}{dt'}\, da = \int_{\partial V} \frac{dW}{dt} \frac{\partial t}{\partial t'}\, da$

$$= \int_{\partial V} \vec{S} \cdot \vec{n}\, \frac{\partial t}{\partial t'}\, da \qquad (22)$$

$t = t' + \frac{1}{c} R$: $\frac{\partial t}{\partial t'} = 1 + \frac{1}{c} \frac{\partial}{\partial t'} R = 1 - \frac{\vec{R}}{R} \cdot \frac{\vec{V}_0}{c}$ (23)

(22) + (23) : $P = \int_{\partial V} (\vec{S} \cdot \vec{n}) \left(1 - \frac{\vec{R}}{R} \cdot \frac{\vec{V}_0}{c}\right) da$ (24)

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) \quad \text{with} \quad \vec{B} = \frac{1}{c} \frac{\vec{R}}{R} \times \vec{E}$$

$$\vec{S} = \frac{1}{\mu_0 c} \vec{E} \times \frac{\vec{R}}{R} \times \vec{E} = \frac{1}{\mu_0 c} \left[\frac{\vec{R}}{R} (\vec{E} \cdot \vec{E}) - \vec{E} \left(\frac{\vec{R}}{R} \cdot \vec{E} \right) \right] \qquad (25)$$

Use radiation field from equation (15) $\rightarrow$ $(\vec{R} \cdot \vec{E}) = 0$!

$$\vec{S} = \frac{1}{\mu_0 c} |\vec{E}|^2 \frac{\vec{R}}{R} \qquad (26)$$

Finally:
$$P = \int_{\partial V} (\vec{S}\cdot\vec{n}) \left(1 - \frac{\vec{R}}{R}\cdot\frac{\vec{v}_0}{c}\right) da$$

$$= \frac{1}{\mu_0 c} \int_{\partial V} \left(|\vec{E}|^2 \frac{\vec{R}}{R}\cdot\vec{n}\right)\left(1 - \frac{\vec{R}}{R}\cdot\frac{\vec{v}_0}{c}\right) da$$

Integrate over spherical surface:

$$P = \frac{1}{\mu_0 c} \int_0^{\pi}\int_0^{2\pi} |\vec{E}|^2 \underbrace{\left(\frac{\vec{R}}{R}\cdot\frac{\vec{R}}{R}\right)}_{1} \left(1 - \frac{\vec{R}}{R}\cdot\frac{\vec{v}_0}{c}\right) R^2 \underbrace{\sin\theta \; d\phi \, d\theta}_{d\Omega \text{ (unit solid angle)}} \tag{27}$$

$$\frac{dP}{d\Omega} = \frac{1}{\mu_0 c} |\vec{E}|^2 R^2 \left(1 - \frac{\vec{R}}{R}\cdot\frac{\vec{v}_0}{c}\right) \tag{28}$$

Insert Eq. (15):

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q^2}{4\pi}\right)\frac{\left[\hat{R}\times(c\hat{R}-\vec{v}_0)\times\vec{a}_0\right]^2}{\left(\hat{R}\cdot[c\hat{R}-\vec{v}_0]\right)^5}} \quad \text{with } \hat{R} = \frac{\vec{R}}{R} \tag{29}$$

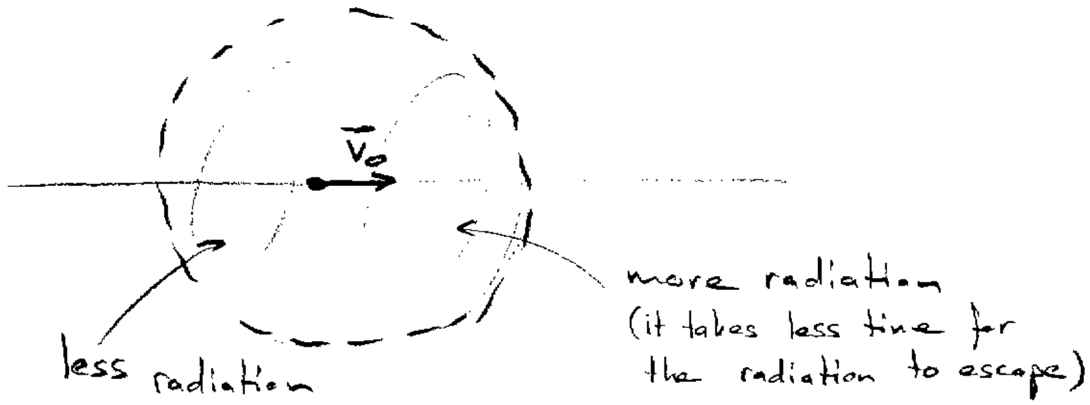
Total radiated power:

$$\boxed{P = \int_{4\pi} \frac{dP}{d\Omega}\, d\Omega = \frac{1}{4\pi\varepsilon_0}\frac{2q^2 a_0^2}{3c^3}\left\{\frac{\left[a_0^2 - \left(\frac{\vec{v}_0}{c}\times\vec{a}_0\right)^2\right]}{\left[1-(v_0/c)^2\right]^3}\right\}} \tag{30}$$

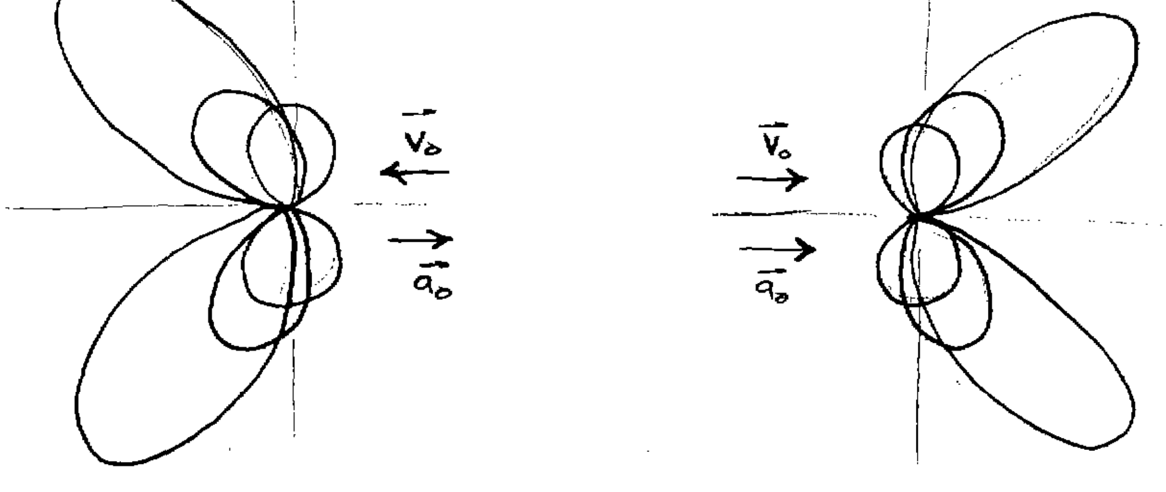
For speeds $v_0 \ll c$, the expression in brackets $\{..\}$ is equal to one. The remaining expression corresponds to <u>Larmor's formula</u>. The latter is identical to the case $dt/dt' = 1$, i.e. $P_s = P$. In this limit, the power radiated by the charge is equal to the power that crosses through the sphere. The correction in the brackets has been derived by Liénard.

Where is the radiation going? → Evaluate Eq. 29

Intuitively:

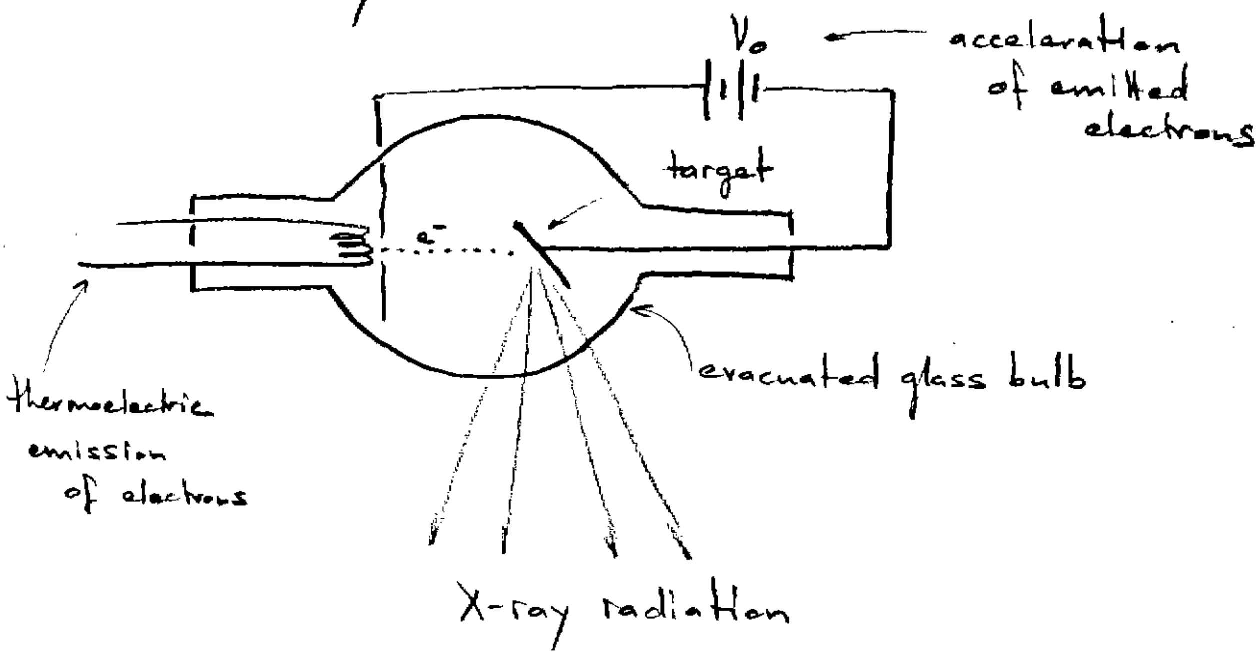


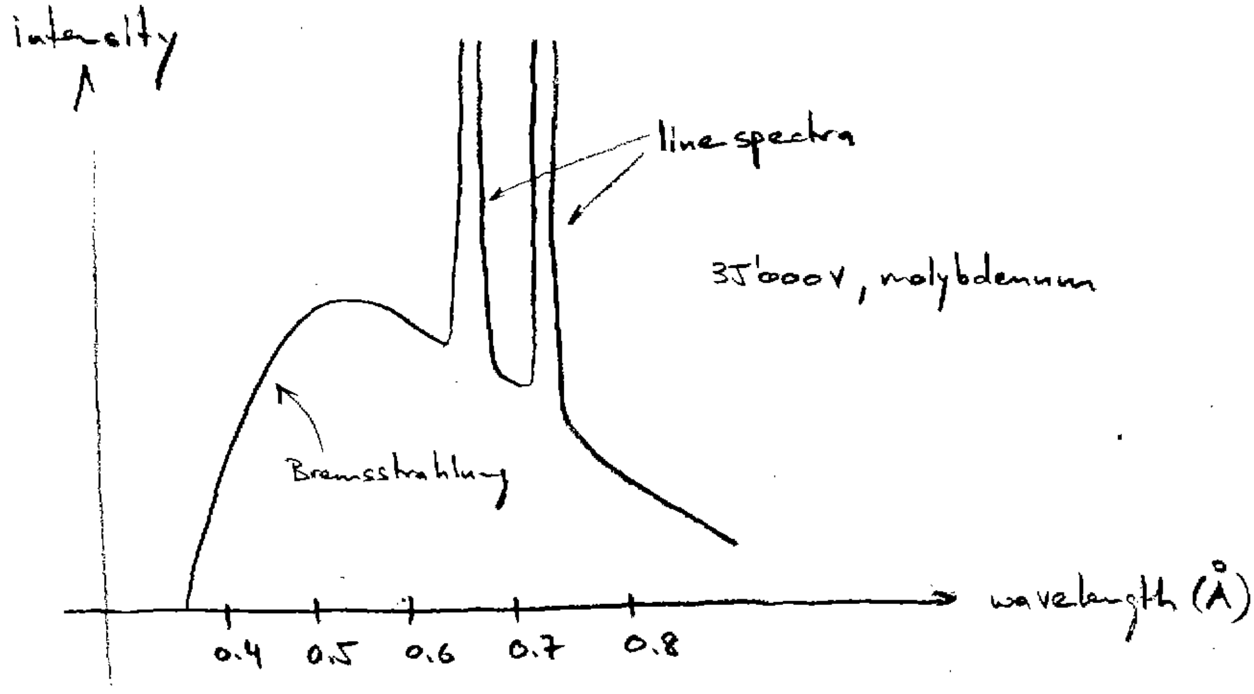
$\vec{v}_0 \parallel \vec{a}_0$: BREMSSTRAHLUNG



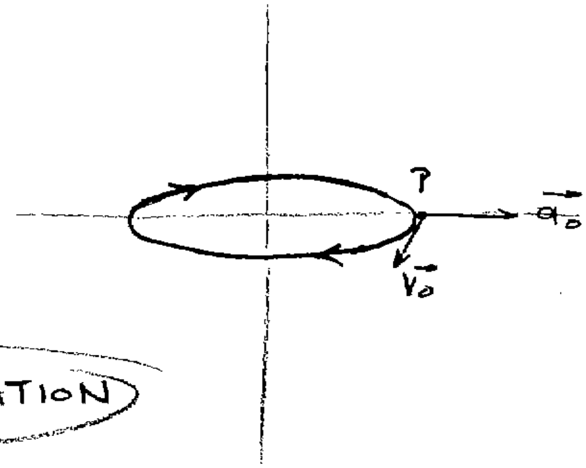
Notice: $P \to \infty$ for $v_0 \to c$! c must be ultimate speed !!!

Observation in X-ray tube:

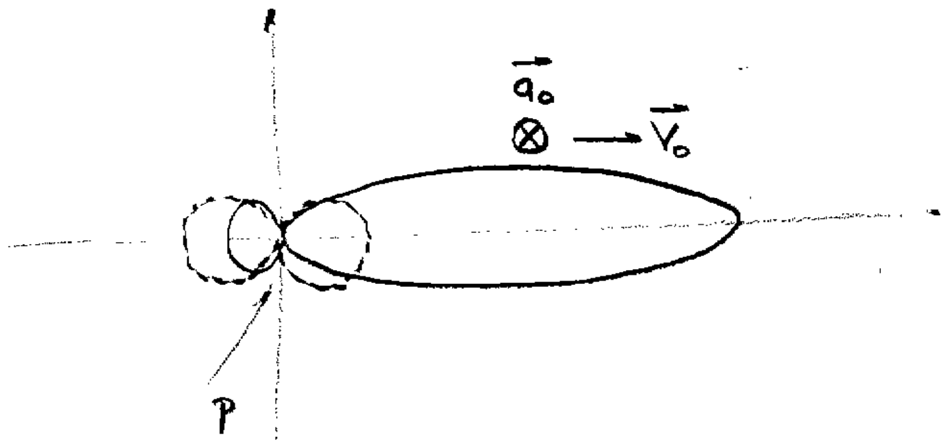


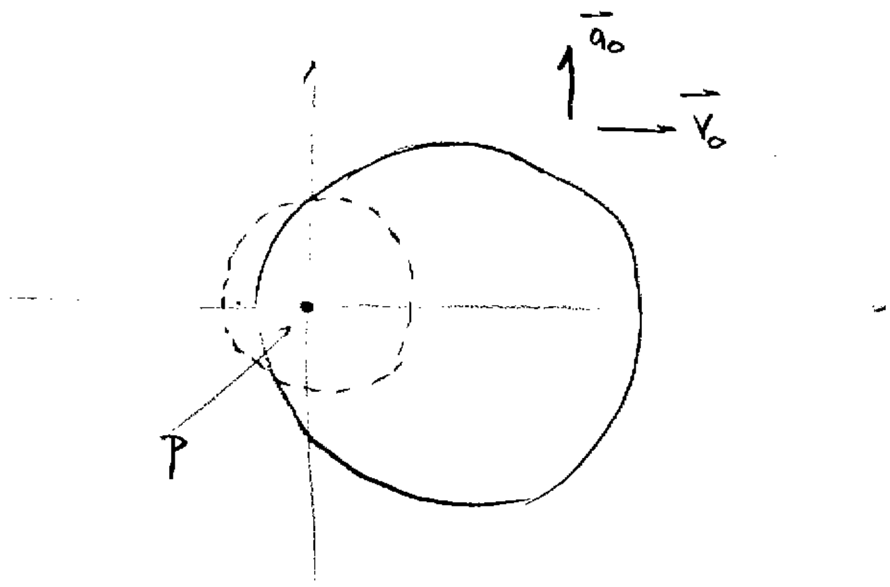


circular motion : $\vec{v}_0 \perp \vec{a}_0$



SYNCHROTRON RADIATION





source for broadband light.