Lagrangian and Hamiltonian of Electron in EM Field

(A) Lagrangian mechanics: $S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$

→ a particle follows the path from time t_1 to time t_2 which minimizes S !
(principle of least action)

→ Lagrange-Euler equation $$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \qquad (1)$$

→ for conservative system (energy preserving):
$$L = T - V$$

(B) Canonical momentum conjugate:

$q_i \hat{=}$ generalized coordinate (e.g. $x, \theta, s, \ldots$)

$p_i = \frac{\partial L}{\partial \dot{q}_i}$ $\hat{=}$ canonical momentum conjugate to q

→ inserting in Eq. (1) we obtain Newton's 2nd law.

(C) Hamiltonian mechanics: $$\boxed{H(q,p) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q})} \qquad (2)$$

→ Hamilton's canonical equations:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \ , \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

→ conservative systems: $H = T + V$

charge;
<u>not</u> generalized coordinate!!

$$-\frac{\partial}{\partial t}\vec{A} - \nabla\phi \qquad \nabla\times\vec{A}$$

$$\rightarrow \quad \frac{d}{dt}[m\dot{x}] = -q\frac{\partial\phi}{\partial x} - q\frac{\partial A_x}{\partial t} + q\left[\frac{\partial y}{\partial t}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial z}{\partial t}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

$$= \frac{\partial}{\partial x}\left[-q\phi + q\left(\frac{\partial x}{\partial t}A_x + \frac{\partial y}{\partial t}A_y + \frac{\partial}{\partial z}A_z\right)\right]$$

$$-q\underbrace{\left[\frac{\partial A_x}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial A_x}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial A_y}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial A_z}{\partial z}\right]}_{= \frac{d}{dt}A_x(\vec{r},t)}$$

rearranging: $\frac{d}{dt}\left[m\frac{\partial x}{\partial t} + qA_x\right] = \frac{\partial}{\partial x}\left[-q\phi + q\left(\frac{\partial x}{\partial t}A_x + \frac{\partial y}{\partial t}A_y + \frac{\partial}{\partial z}A_z\right)\right]$

Comparison with Eq. (1): (left hand side)

a) $L = -q\phi + q\left(\frac{\partial x}{\partial t}A_x + \frac{\partial y}{\partial t}A_y + \frac{\partial z}{\partial t}A_z\right) + f(\ldots)$

where $\frac{\partial f}{\partial x} = 0$!!

b) $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] = \frac{d}{dt}\left[qA_x + \frac{\partial f}{\partial \dot{x}}\right]$

c) comparison with right hand side: $\frac{\partial f}{\partial \dot{x}} = m\frac{\partial x}{\partial t}$

$\rightarrow f = \frac{1}{2}m\dot{x}^2$

(E) Repeating for y-component of $\vec{F}$ and z-component:

$$\boxed{L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\phi + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)}$$

or

$$L = \frac{1}{2} m \vec{v}\cdot\vec{v} - q\phi + q\vec{v}\cdot\vec{A}$$

$\rightarrow$ independent of gauge !!

(F) Canonical momentum conjugate:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x \longrightarrow \boxed{\vec{p} = \underbrace{m\vec{v}}_{\text{mechanical momentum}} + \underbrace{q\vec{A}}_{\text{field momentum}}} \qquad (4)$$

The time-dependent electromagnetic field is not conservative
The energy it takes for a particle to go from A to B is
path dependent !! $\longrightarrow$ $L \neq T - V$

(G) Hamiltonian:

General rule $\rightarrow$ express $\dot{q}_i$ in terms of q_i and p_i

According Eq. (4): $\dot{x} = \frac{p_x}{m} - \frac{q}{m}A_x$, $\dot{y} = \frac{p_y}{m} - \frac{q}{m}A_y$, $\dot{z} = \frac{p_z}{m} - \frac{q}{m}A_z$

$$\rightarrow L(p,q) = \frac{1}{2m}\left[|\vec{p}|^2 - q^2|\vec{A}|^2\right] - q\phi \qquad (5)$$

$$\sum_{i=1}^{3} p_i \dot{q}_i = \sum_{i=1}^{3} p_i\left(\frac{p_i}{m} - \frac{q}{m} A_i\right) = \frac{|\vec{p}|^2}{m} - \frac{q}{m}\vec{p}\cdot\vec{A} \qquad (6)$$

Inserting Eq.(5) and Eq.(6) into Eq.(2):

$$\boxed{H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi} \qquad (7)$$

The Hamiltonian is <u>not</u> of the form $H = T + V$!

CAREFUL: Some texts use different definition of $\vec{A}$, i.e. $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}$. The Hamiltonian then reads as

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$