

Lagrangian and Hamiltonian of Electron in EM Field

①

(A) Lagrangian mechanics: $S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$

→ a particle follows the path from time t_1 to time t_2 which minimizes S !
(principle of least action)

→ Lagrange-Euler equation $\left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \right]$ (1)

→ for conservative system (energy preserving):
 $L = T - V$

(B) Canonical momentum conjugate:

$q_i \hat{=}$ generalized coordinate (e.g. x, θ, S, \dots)

$p_i = \frac{\partial L}{\partial \dot{q}_i} \hat{=}$ canonical momentum conjugate to q

→ inserting in Eq. (1) we obtain Newton's 2nd law.

(C) Hamiltonian mechanics: $\left[H(q, p) = \sum_{i=1}^n p_i \dot{q}_i - L(q, \dot{q}) \right]$ (2)

→ Hamilton's canonical equations:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

→ conservative systems: $H = T + V$

charge;
not generalized
coordinate !!

$$-\frac{\partial \vec{A}}{\partial t} - \nabla \phi$$

$$\nabla \times \vec{A}$$

$$\begin{aligned} \rightarrow \frac{d}{dt} [m\dot{x}] &= -q \frac{\partial \phi}{\partial x} - q \frac{\partial A_x}{\partial t} + q \left[\frac{\partial y}{\partial t} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \frac{\partial z}{\partial t} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] \\ &= \frac{\partial}{\partial x} \left[-q\phi + q \left(\frac{\partial x}{\partial t} A_x + \frac{\partial y}{\partial t} A_y + \frac{\partial z}{\partial t} A_z \right) \right] \\ &\quad - q \left[\frac{\partial A_x}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial A_x}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial A_y}{\partial y} + \frac{\partial z}{\partial t} \frac{\partial A_z}{\partial z} \right] \\ &= \frac{d}{dt} A_x(\vec{r}, t) \end{aligned}$$

$$\text{rearranging: } \frac{d}{dt} \left[m \frac{\partial x}{\partial t} + q A_x \right] = \frac{\partial}{\partial x} \left[-q\phi + q \left(\frac{\partial x}{\partial t} A_x + \frac{\partial y}{\partial t} A_y + \frac{\partial z}{\partial t} A_z \right) \right]$$

Comparison with Eq. (1): (left hand side)

$$a) \quad L = -q\phi + q \left(\frac{\partial x}{\partial t} A_x + \frac{\partial y}{\partial t} A_y + \frac{\partial z}{\partial t} A_z \right) + f(\dots)$$

$$\text{where } \frac{\partial f}{\partial x} = 0 \quad !!$$

$$b) \quad \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{x}} \right] = \frac{d}{dt} \left[q A_x + \frac{\partial f}{\partial \dot{x}} \right]$$

$$c) \quad \text{comparison with right hand side: } \frac{\partial f}{\partial \dot{x}} = m \frac{\partial x}{\partial t}$$

$$\rightarrow f = \frac{1}{2} m \dot{x}^2$$

(E) Repeating for y-component of \vec{F} and z-component:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\phi + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)$$

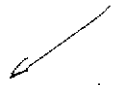
or

$$L = \frac{1}{2} m \vec{v} \cdot \vec{v} - q\phi + q \vec{v} \cdot \vec{A}$$


→ independent of gauge !!

(F) Canonical momentum conjugate:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x \quad \rightarrow \quad \boxed{\vec{p} = m\vec{v} + q\vec{A}} \quad (4)$$



mechanical
momentum



field momentum

The time-dependent electromagnetic field is not conservative
The energy it takes for a particle to go from A to B is path dependent !! → $L \neq T - V$

(G) Hamiltonian:

General rule → express \dot{q}_i in terms of q_i and p_i

According Eq. (4): $\dot{x} = \frac{p_x}{m} - \frac{q}{m} A_x$, $\dot{y} = \frac{p_y}{m} - \frac{q}{m} A_y$, $\dot{z} = \frac{p_z}{m} - \frac{q}{m} A_z$

$$\rightarrow L(p, q) = \frac{1}{2m} [|\vec{p}|^2 - q^2 |\vec{A}|^2] - q\phi \quad (5)$$

$$\sum_{i=1}^3 p_i \dot{q}_i = \sum_{i=1}^3 p_i \left(\frac{p_i}{m} - \frac{q}{m} A_i \right) = \frac{|\vec{p}|^2}{m} - \frac{q}{m} \vec{p} \cdot \vec{A} \tag{6}$$

Inserting Eq.(5) and Eq.(6) into Eq.(2):

$$\boxed{H = \frac{1}{2m} (\vec{p} - q\vec{A})^2 + q\phi} \tag{7}$$

The Hamiltonian is not of the form $H = T + V$!

CAREFUL: Some texts use different definition of \vec{A} ,
 i.e. $\vec{E} = -\nabla\phi - \frac{1}{c} \frac{\partial}{\partial t} \vec{A}$. The Hamiltonian then
 reads as

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 + q\phi$$