TIME DEPENDENT GREEN'S FUNCTIONS

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G_0(\vec{r},\vec{r}';t,t') = -\delta(\vec{r}-\vec{r}')\,\delta(t-t') \qquad \text{(vacuum)} \tag{1}$$

homogeneous space + time: $G_0(\vec{r},\vec{r}';t,t') = G_0(|\vec{r}-\vec{r}'|;t-t') = G_0(R;\tau)$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right] G_0(R;\tau) = -\delta(R)\,\delta(\tau) \tag{2}$$

Fourier transform:

$$\underbrace{\int_{-\infty}^{\infty}\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2}\right] G_0(R;\tau)\, e^{i\omega\tau}\,d\tau}_{\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\underbrace{\int_{-\infty}^{\infty} G_0(R;\tau)\, e^{i\omega\tau}\,d\tau}_{=\hat{G}_0(R;\omega)}} = -\delta(R)\underbrace{\int_{-\infty}^{\infty}\delta(\tau)\, e^{i\omega\tau}\,d\tau}_{=1}$$

$$\rightarrow \left[\nabla^2 + \frac{\omega^2}{c^2}\right]\hat{G}_0(R;\omega) = -\delta(R) \qquad \text{where } R = |\vec{r}-\vec{r}'|$$

In homogeneous, isotropic, linear medium with $n(\omega) = \sqrt{\varepsilon(\omega)\mu(\omega)}$:

$$\boxed{\left[\nabla^2 + k^2(\omega)\right]\hat{G}_0(R;\omega) = -\delta(R)} \qquad \text{where} \quad k(\omega) = \frac{\omega}{c}\,n(\omega) \tag{3}$$

Backtransform:

$$G_0(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{G}_0(R,\omega)\, e^{-i\omega\tau}\,d\omega \tag{4}$$

Solution of (4) : $\hat{G}_o(R;\omega) = \frac{1}{4\pi} \frac{e^{\pm i k(\omega) R}}{R}$

time-dependent solution: $G_o(R;\tau) = \frac{1}{8\pi^2 R} \int_{-\infty}^{\infty} e^{\pm i \frac{\omega}{c} n(\omega) R} \, e^{-i\omega\tau} \, d\omega$

no dispersion : $n(\omega) = n \qquad \int_{-\infty}^{\infty} e^{-i\omega(\tau \mp \frac{n}{c} R)} \, d\omega = 2\pi \, \delta(\tau \mp \frac{n}{c} R)$

$$\boxed{G_o(R;\tau) = \frac{\delta(\tau \mp \frac{n}{c} R)}{4\pi R}} \tag{5}$$

causality : $\tau = t - t' \quad \rightarrow \quad G_o(\vec{r},\vec{r}\,';t,t') = \frac{\delta\left[t' - (t - \frac{n}{c}|\vec{r}-\vec{r}\,'|)\right]}{4\pi \, |\vec{r}-\vec{r}\,'|}$ (6)

RETARDED GREEN'S FUNCTION

The '+' sign would imply that G_o at time $t' = t$ is generated by the future time $(t + \frac{n}{c} R)$, which has to be rejected on basis of causality arguments.

SOLUTION OF $\hat{G}_0(R;\omega)$

$$\left[\nabla^2 + k^2\right]\hat{G}_0 = -\delta(R)$$

→ spherical coordinates : $\hat{G}_0 = \hat{G}_0(R,\theta,\phi)$

→ since homogeneous space : $\hat{G}_0 = \hat{G}_0(R)$

$$\nabla^2 = \frac{1}{R}\frac{\partial^2}{\partial R^2}(R\hat{G}_0)$$

Solution of homogeneous equation: $(R\hat{G}_0) = A\,e^{ikR} + B\,e^{-ikR}$ (1)

Solution at $R \to 0$: $\lim_{R\to 0} \hat{G}_0 = \frac{1}{R}(A+B)$ (2)

Consider small volume ΔV (spherical with radius R_0) :

$$\int_{\Delta V} [\nabla^2 + k^2]\left(\frac{A+B}{R}\right) dV = -\int_{\Delta V} \delta(R)\, dV = -1$$

$$\int_{\Delta V} \nabla^2\left(\frac{1}{R}\right) dV + \underbrace{k^2 \int_{\Delta V} \frac{1}{R}\, dV}_{2\pi (kR_0)^2} = -\frac{1}{A+B} \qquad (3)$$

$$\int_{\Delta V} \nabla\cdot\left(\nabla \frac{1}{R}\right) dV = \int_{\partial \Delta V} \nabla\left(\frac{1}{R}\right)\cdot \vec{n}\, da = -\int_{\partial \Delta V} \frac{1}{R^3}\vec{R}\cdot\vec{n}\, da = -4\pi$$

Thus $\lim_{\Delta V \to 0} \int_{\Delta V} [\nabla^2 + k^2]\hat{G}_0\, dV = -4\pi$ → $A + B = \frac{1}{4\pi}$

OUTGOING WAVES : $A = 0$

$$\longrightarrow \boxed{\hat{G}_0(\vec{r},\vec{r}\,';\omega) = \frac{e^{-ik(\omega)|\vec{r}-\vec{r}\,'|}}{4\pi\,|\vec{r}-\vec{r}\,'|}} \qquad (4)$$