

1.3.10 Dyadic Green's functions

An important concept in field theory are Green's functions: the fields due to a point source. In electromagnetic theory, the dyadic Green's function $\vec{\mathbf{G}}$ is essentially defined by the electric field \mathbf{E} at the fieldpoint \mathbf{r} generated by a radiating electric dipole \mathbf{p} located at the source point \mathbf{r}' . In mathematical terms this reads as

$$\mathbf{E}(\mathbf{r}) = \omega^2 \mu_o \mu \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{p}(\mathbf{r}') . \quad (1.51)$$

To understand the basic idea of Green's functions we will first consider a general mathematical point of view.

Mathematical basis of Green's functions

Consider the following general, inhomogeneous equation:

$$\mathcal{L} \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r}) . \quad (1.52)$$

\mathcal{L} is a linear operator acting on the vectorfield \mathbf{A} representing the unknown response of the system. The vectorfield \mathbf{B} is a known source function and makes the differential equation inhomogeneous. A well known theorem for linear differential equations states that the general solution is equal to the sum of the complete homogeneous solution ($\mathbf{B}=0$) and a particular inhomogeneous solution. Here, we assume that the homogeneous solution (\mathbf{A}_o) is known. We thus need to solve for an arbitrary particular solution.

Usually it is difficult to find a solution of Eq. 1.52 and it is easier to consider the special inhomogeneity $\delta(\mathbf{r}-\mathbf{r}')$, which is zero everywhere, except in the point $\mathbf{r}=\mathbf{r}'$. Then, the linear equation reads as

$$\mathcal{L} \mathbf{G}_i(\mathbf{r}, \mathbf{r}') = \mathbf{n}_i \delta(\mathbf{r} - \mathbf{r}') \quad (i=x, y, z) , \quad (1.53)$$

where \mathbf{n}_i denotes an arbitrary constant unit vector. In general, the vectorfield \mathbf{G}_i is dependent on the location \mathbf{r}' of the inhomogeneity $\delta(\mathbf{r}-\mathbf{r}')$. Therefore, the vector \mathbf{r}' has been included in the argument of \mathbf{G}_i . The three equations given by Eq. 1.53 can be written in closed form as

$$\mathcal{L} \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \vec{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}') , \quad (1.54)$$

where the operator \mathcal{L} acts on each column of $\vec{\mathbf{G}}$ separately and $\vec{\mathbf{I}}$ is the unit dyad. The dyadic function $\vec{\mathbf{G}}$ fulfilling Eq. 1.54 is known as the dyadic Green's function.

In a next step, assume that Eq. 1.54 has been solved and that $\vec{\mathbf{G}}$ is known. Postmultiplying Eq. 1.54 with $\mathbf{B}(\mathbf{r}')$ on both sides and integrating over the volume V in which $\mathbf{B} \neq 0$ gives

$$\int_V \mathcal{L} \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{B}(\mathbf{r}') dV' = \int_V \mathbf{B}(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') dV' . \quad (1.55)$$

The right hand side simply reduces to $\mathbf{B}(\mathbf{r})$ and with Eq. 1.52 it follows that

$$\mathcal{L} \mathbf{A}(\mathbf{r}) = \int_V \mathcal{L} \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{B}(\mathbf{r}') dV' . \quad (1.56)$$

If on the right hand side the operator \mathcal{L} is taken out of the integral, the solution of Eq. 1.52 can be expressed as

$$\mathbf{A}(\mathbf{r}) = \int_V \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{B}(\mathbf{r}') dV'. \quad (1.57)$$

Thus, the solution of the original equation can be found by integrating the product of the dyadic Green's function and the inhomogeneity \mathbf{B} over the source volume V .

The assumption that the operators \mathcal{L} and $\int dV'$ can be interchanged is not strictly valid and special care must be applied if the integrand is not well behaved. Most often $\vec{\mathbf{G}}$ is singular at $\mathbf{r} = \mathbf{r}'$ and an infinitesimal exclusion volume surrounding $\mathbf{r} = \mathbf{r}'$ has to be introduced [28, 29]. Depolarization of the principal volume must be treated separately resulting in a term ($\vec{\mathbf{L}}$) depending on the geometrical shape of the volume. Furthermore, in numerical schemes the principal volume has a finite size giving rise to a second correction term designated by $\vec{\mathbf{M}}$. As long as we consider field points outside of the source volume V , i.e., $\mathbf{r} \notin V$ we do not need to consider these tricky issues. However, the topic of the principal volume will be taken up in later chapters.

Derivation of the Green's function for the electric field

The derivation of the Green's function for the electric field is most conveniently accomplished by considering the time-harmonic vector potential \mathbf{A} and the scalar potential ϕ in an infinite and homogeneous space which is characterized by the constants ε and μ . In this case, \mathbf{A} and ϕ are defined by the relationships

$$\mathbf{E}(\mathbf{r}) = i\omega \mathbf{A}(\mathbf{r}) - \nabla\phi(\mathbf{r}) \quad (1.58)$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_o\mu} \nabla \times \mathbf{A}(\mathbf{r}). \quad (1.59)$$

We can insert these equations into Maxwell's second equation Eq. 1.2 and obtain

$$\nabla \times \nabla \times \mathbf{A}(\mathbf{r}) = \mu_o\mu\mathbf{j}(\mathbf{r}) - i\omega\mu_o\mu\varepsilon_o\varepsilon [i\omega\mathbf{A}(\mathbf{r}) - \nabla\phi(\mathbf{r})], \quad (1.60)$$

where we used $\mathbf{D} = \varepsilon_o\varepsilon\mathbf{E}$. The potentials \mathbf{A}, ϕ are not uniquely defined by Eqs. 1.58, 1.59. We are still free to define the value of $\nabla \cdot \mathbf{A}$ which we choose as

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = i\omega\mu_o\mu\varepsilon_o\varepsilon\phi(\mathbf{r}). \quad (1.61)$$

A condition which fixes the redundancy of Eqs. 1.58, 1.59 is called a gauge condition. The gauge chosen through Eq. 1.61 is the so-called Lorentz gauge. Using the mathematical identity ' $\nabla \times \nabla \times = -\nabla^2 + \nabla\nabla \cdot$ ' together with the Lorentz gauge we can rewrite Eq. 1.60 as

$$[\nabla^2 + k^2] \mathbf{A}(\mathbf{r}) = -\mu_o\mu\mathbf{j}(\mathbf{r}), \quad (1.62)$$

which is the inhomogeneous Helmholtz equation. It holds independently for each component A_i of \mathbf{A} . A similar equation can be derived for the scalar potential ϕ

$$[\nabla^2 + k^2] \phi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_o\varepsilon. \quad (1.63)$$

Thus, we obtain four scalar Helmholtz equations of the form

$$[\nabla^2 + k^2]f(\mathbf{r}) = -g(\mathbf{r}) . \quad (1.64)$$

To derive the *scalar* Green's function $G_o(\mathbf{r}, \mathbf{r}')$ for the Helmholtz operator we replace the source term $g(\mathbf{r})$ by a single point source $\delta(\mathbf{r}-\mathbf{r}')$ and obtain

$$[\nabla^2 + k^2]G_o(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') . \quad (1.65)$$

The coordinate \mathbf{r} denotes the location of the field point, i.e. the point in which the fields are to be evaluated, whereas the coordinate \mathbf{r}' designates the location of the point source. Once we have determined G_o we can state the particular solution for the vector potential in Eq. 1.62 as

$$\mathbf{A}(\mathbf{r}) = \mu_o\mu \int_V \mathbf{j}(\mathbf{r}') G_o(\mathbf{r}, \mathbf{r}') dV' . \quad (1.66)$$

A similar equation holds for the scalar potential. Both solutions require the knowledge of the Green's function defined through Eq. 1.65. In free space, the only physical solution of this equation is [30]

$$G_o(\mathbf{r}, \mathbf{r}') = \frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} . \quad (1.67)$$

The solution with the plus sign denotes a spherical wave that propagates out of the origin whereas the solution with the minus sign is a wave that converges towards the origin. In the following we only retain the outwards propagating wave. The scalar Green's function can be introduced into Eq. 1.66 and the vector potential can be calculated by integrating over the source volume V . Thus, we are in a position to calculate the vector potential and scalar potential for any given current distribution \mathbf{j} and charge distribution ρ . Notice, that the Green's function in Eq. 1.67 applies only to a homogeneous three-dimensional space. The Green's function of a two-dimensional space or a half-space will have a different form.

So far we reduced the treatment of Green's functions to the potentials \mathbf{A} and ϕ because it allows us to work with scalar equations. The formalism becomes more involved when we consider the electric and magnetic fields. The reason for this is that a source current in x -direction leads to an electric and magnetic field with x , y , and z -components. This is different for the vector potential: a source current in x gives only rise to a vector potential with a x component. Thus, in the case of the electric and magnetic fields we need a Green's function which relates all components of the source with all components of the fields, or, in other words, the Green's function must be a tensor. This type of Green's function is denoted as *dyadic Green's function* and has been introduced in the previous section. To determine the dyadic Green's function we start with the waveequation for the electric field Eq. 1.31. In a homogeneous space it reads as

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2 \mathbf{E}(\mathbf{r}) = i\omega\mu_o\mu\mathbf{j}(\mathbf{r}) . \quad (1.68)$$

We can define for each component of \mathbf{j} a corresponding Green's function. For example, for j_x we have

$$\nabla \times \nabla \times \mathbf{G}_x(\mathbf{r}, \mathbf{r}') - k^2 \mathbf{G}_x(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\mathbf{n}_x , \quad (1.69)$$

where \mathbf{n}_x is the unit vector in x -direction. A similar equation can be formulated for a point source in y and z -direction. In order to account for all orientations we write as the general definition of the dyadic Green's function for the electric field [32]

$$\nabla \times \nabla \times \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') - k^2 \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \vec{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}') , \quad (1.70)$$

$\vec{\mathbf{I}}$ being the unit dyad (unit tensor). The first column of the tensor $\vec{\mathbf{G}}$ corresponds to the field due to a point source in x -direction, the second column to the field due to a point source in y -direction, and the third column is the field due to a point source in z -direction. Thus a dyadic Green's function is just a compact notation for three vectorial Green's functions.

As before, we can view the source current in Eq. 1.68 as a superposition of point currents. Thus, if we know the Greens' function $\vec{\mathbf{G}}$ we can state a particular solution of Eq. 1.68 as

$$\mathbf{E}(\mathbf{r}) = i\omega\mu\mu_o \int_V \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{j}(\mathbf{r}') dV' . \quad (1.71)$$

However, this is a particular solution and we need to add any homogeneous solutions \mathbf{E}_o . Thus, the general solution turns out to be

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_o(\mathbf{r}) + i\omega\mu\mu_o \int_V \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{j}(\mathbf{r}') dV' \quad \mathbf{r}' \notin V . \quad (1.72)$$

The corresponding magnetic field reads as

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_o(\mathbf{r}) + \int_V [\nabla \times \vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}')] \mathbf{j}(\mathbf{r}') dV' \quad \mathbf{r}' \notin V . \quad (1.73)$$

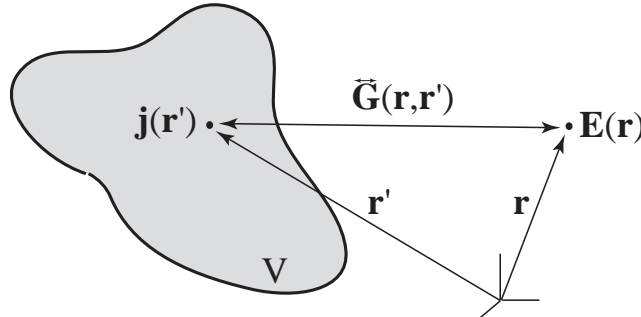


Figure 1.5: Illustration of the dyadic Green's function $\vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$. The Green's function renders the electric field at the field point \mathbf{r} due to a single point source \mathbf{j} at the source point \mathbf{r}' . Since the field at r depends on the orientation of \mathbf{j} the Green's function must account for all possible orientations in the form of a tensor.

These equations are denoted as *volume integral equations*. They are very important since they form the basis for various formalisms such as the ‘method of moments’, the ‘Lippmann-Schwinger equation’, or the ‘coupled dipole method’. We have limited the validity of the volume integral equations to the space outside the source volume V in order to avoid the apparent singularity of $\vec{\mathbf{G}}$ at $\mathbf{r} = \mathbf{r}'$. This limitation will be relaxed in later chapters.

In order to solve Eqs. 1.72 and 1.73 for a given distribution of currents, we still need to determine the value of $\vec{\mathbf{G}}$. Introducing the Lorentz gauge Eq. 1.61 into Eq. 1.58 leads to

$$\mathbf{E}(\mathbf{r}) = i\omega \left[1 + \frac{1}{k^2} \nabla \nabla \cdot \right] \mathbf{A}(\mathbf{r}) . \quad (1.74)$$

The first column vector of $\vec{\mathbf{G}}$, i.e. \mathbf{G}_x , defined in Eq. 1.69 is simply the electric field due to a point source current $\mathbf{j} = (i\omega\mu_o)^{-1} \delta(\mathbf{r} - \mathbf{r}') \mathbf{n}_x$. The vector potential originating from this source current is according to Eq. 1.66

$$\mathbf{A}(\mathbf{r}) = (i\omega)^{-1} G_o(\mathbf{r}, \mathbf{r}') \mathbf{n}_x . \quad (1.75)$$

Inserting this vector potential into Eq. 1.74 we find

$$\mathbf{G}_x(\mathbf{r}, \mathbf{r}') = \left[1 + \frac{1}{k^2} \nabla \nabla \cdot \right] G_o(\mathbf{r}, \mathbf{r}') \mathbf{n}_x , \quad (1.76)$$

with similar expressions for \mathbf{G}_y and \mathbf{G}_z . The only remaining thing to do is to tie the three solutions together to form a dyad. With the definition $\nabla \cdot [G_o \vec{\mathbf{I}}] = \nabla G_o$ the dyadic Green’s function $\vec{\mathbf{G}}$ can be calculated from the scalar Green’s function G_o in Eq. 1.67 as

$$\boxed{\vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \left[\vec{\mathbf{I}} + \frac{1}{k^2} \nabla \nabla \right] G_o(\mathbf{r}, \mathbf{r}')} . \quad (1.77)$$

Time dependent Green’s functions

The time dependence in the wave equations can be separated and the resulting harmonic differential equation for the time behavior is easily solved. A monochromatic field can be represented in the form of Eq. 1.24 and any other time-dependent field can be generated by a Fourier transform (sum of monochromatic fields). However, for the study of ultrafast phenomena it is of advantage to retain the explicit time behavior. In this case we have to generalize the definition of \mathbf{A} and ϕ as⁴

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) - \nabla \phi(\mathbf{r}, t) \quad (1.78)$$

$$\mathbf{H}(\mathbf{r}, t) = \frac{1}{\mu_o \mu} \nabla \times \mathbf{A}(\mathbf{r}, t) , \quad (1.79)$$

⁴We assume a nondispersive medium, i.e. $\varepsilon(\omega) = \varepsilon$ and $\mu(\omega) = \mu$.

from which we find the time-dependent Helmholtz equation in the Lorentz gauge (c.f. Eq. 1.62)

$$\left[\nabla^2 - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{A}(\mathbf{r}, t) = -\mu_o \mu \mathbf{j}(\mathbf{r}, t). \quad (1.80)$$

A similar equation holds for the scalar potential ϕ . The definition of the *scalar* Green's function is now generalized to

$$\left[\nabla^2 - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \right] G_o(\mathbf{r}, \mathbf{r}'; t, t') = -\delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \quad (1.81)$$

The point source is now defined with respect to space and time. The solution for G_o is [30]

$$G_o(\mathbf{r}, \mathbf{r}'; t, t') = \frac{\delta\left(t' - \left[t \mp \frac{n}{c} |\mathbf{r} - \mathbf{r}'|\right]\right)}{4\pi |\mathbf{r} - \mathbf{r}'|}, \quad (1.82)$$

where the minus sign is associated with the response at a time t later than t' . Using G_o we can construct the time-dependent dyadic Green's function $\vec{\mathbf{G}}(\mathbf{r}, \mathbf{r}'; t, t')$ similar to the previous case. Since we will mostly work with time-independent Green's functions we avoid further details and refer the interested reader to specialized books on electrodynamics. Working with time-dependent Green's functions accounts for arbitrary time behavior but it is very difficult to incorporate dispersion. Time dependent processes in dispersive media are more conveniently solved using Fourier transforms of monochromatic fields.