VECTOR AND SCALAR POTENTIALS $(\vec{A}, \phi)$

$\nabla \cdot \vec{B} = 0$ → mathematical identity: $\nabla \cdot (\nabla \times \text{anyfield}) = 0$

→ define: $\boxed{\vec{B} = \nabla \times \vec{A}}$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ → $\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$

→ similar to electrostatics, where $\nabla \times \vec{E} = 0$

$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$ → mathematical identity: $\nabla \times (\nabla \text{ any potential}) = 0$

→ define: $\left(\vec{E} + \frac{\partial}{\partial t}\vec{A}\right) = -\nabla \phi$

all together: $\boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi}$

HERTZ VECTOR $(\vec{\Pi})$

$\vec{A} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{\Pi}$, $\phi = -\nabla \cdot \vec{\Pi}$ (equivalent to $\vec{A}, \phi$ in Lorentz gauge
→ see later)

GAUGE TRANSFORMATIONS

Define $\vec{A}' = \vec{A} + \nabla f$ where f = arbitrary function

Determine fields: × $\vec{B} = \nabla \times \vec{A}' = \nabla \times \vec{A} + \underbrace{\nabla \times \nabla f}_{\equiv 0 \text{ (math. identity)}}$

$$= \nabla \times \vec{A}$$

→ thus: $\vec{A}$ and $\vec{A} + \nabla f$ render the <u>same</u> $\vec{B}$ field!

× $\vec{E} = -\frac{\partial \vec{A}'}{\partial t} - \nabla \phi' = -\frac{\partial}{\partial t}\vec{A} - \frac{\partial}{\partial t}\nabla f - \nabla \phi'$.

→ thus: $\vec{A}$ and $\vec{A} + \nabla f$ render the <u>same</u> $\vec{E}$ field
provided that $\phi' = \phi - \frac{\partial}{\partial t} f$!

$\vec{E}$ and $\vec{B}$ are unchanged by the transformation:

$$\boxed{\begin{aligned} \vec{A} &\rightarrow \vec{A} + \nabla f \\ \phi &\rightarrow \phi - \frac{\partial}{\partial t} f \end{aligned}}$$

GAUGE TRANSFORMATION

Any field f can be 'added' to $\vec{A}, \phi$ according to gauge transformation and it will not alter the fields $\vec{E}$ and $\vec{B}$.

SPECIFICATION OF $\nabla\cdot\vec{A}$ (THE GAUGES)

Any vector field $\vec{F}$ is specified by the definition of

$$\nabla\cdot\vec{F} \quad \text{and} \quad \nabla\times\vec{F}\,.$$

A vector field $\vec{F}$ with $\nabla\cdot\vec{F}=0$ is called <u>transverse</u>, whereas $\nabla\times\vec{F}=0$ defines a <u>longitudinal</u> field.

So far, we have defined the curl of the vector potential, i.e. $\nabla\times\vec{A}=\vec{B}$. However, we did not specify $\nabla\cdot\vec{A}$. The relations between $\vec{A},\phi$ and $\vec{E},\vec{B}$ are independent of the choice of $\nabla\cdot\vec{A}$!!

Microscopic Maxwell equations: $\nabla\cdot\vec{E}=\rho/\varepsilon_0 \;\rightarrow\; \nabla\cdot\left(-\frac{\partial}{\partial t}\vec{A}-\nabla\phi\right)=\rho/\varepsilon_0 \quad (1)$

$$\nabla\times\vec{B}=\mu_0\vec{j}+\frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}$$

$$\rightarrow \nabla\times\nabla\times\vec{A}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}+\frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t}=\mu_0\vec{j} \quad (2)$$

Rewrite last equation:

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}=-\mu_0\vec{j}+\nabla\left[\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right] \quad (3)$$

<u>LORENTZ GAUGE</u>: $\boxed{\nabla\cdot\vec{A}=-\frac{1}{c^2}\frac{\partial}{\partial t}\phi}$ (c.f. $\nabla\cdot\vec{j}=-\partial/\partial t\,\rho$) (4)

Equations (1) and (2) become: $\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{A}=-\mu_0\vec{j} \quad (5)$

$$\left[\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi=-\rho/\varepsilon_0 \quad (6)$$

2 decoupled partial differential equations.

QUESTION: What gauge transformation leads to the Lorentz gauge?
→ determine the condition for the function f

other convenient choices of $\nabla \cdot \vec{A}$:

COULOMB GAUGE : $\boxed{\nabla \cdot \vec{A} = 0}$ (7)

(also called: ~~radiation gauge~~, transverse gauge
minimal coupling gauge)

Eq. (1) reduces to $\nabla^2 \phi = -\rho/\varepsilon_0$ (POISSON EQUATION)

solution: $\phi = \int_{V'} \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \, d\vec{r}\,'$

$\phi \mathrel{\hat{=}}$ instantaneous Coulomb potential

Eq. (2) gives $\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} = -\mu_0 \vec{j} + \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t}$

split $\vec{j}$ into longitudinal and transverse part as
$\vec{j} = \vec{j}_L + \vec{j}_T$, where $\nabla \times \vec{j}_L = 0$ and $\nabla \cdot \vec{j}_T = 0$

$$\longrightarrow \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} = \mu_0 \vec{j}_L \tag{8}$$

$$\longrightarrow \left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \vec{A} = -\mu_0 \vec{j}_T \tag{9}$$

Radiation fields are transverse fields. In the farfield, the scalar potential ϕ can be ignored and all fields are defined by Eq. (9). Notice the similarity with Eq. (5), where $\vec{j} = \vec{j}_L + \vec{j}_T$. Optical near-fields need to include ϕ and $\vec{j}_L$!!