1.3 Macroscopic Electrodynamics

Light embraces the most fascinating spectrum of electromagnetic radiation. This is mainly due to the fact that the energy of light quanta (photons) lie in the energy range of electronic transitions in matter. This gives us the beauty of color and is the reason why our eyes adapted to sense the optical spectrum.

Light is also fascinating because it manifests itself in forms of waves and particles. In no other range of the electromagnetic spectrum we are more confronted with the wave-particle duality than in the optical regime. While long wavelength radiation (radiofrequencies, microwaves) is well described by wave theory, short wavelength radiation (X-rays) exhibits mostly particle properties. The two worlds meet in the optical regime.

To describe optical radiation in nano-optics it is mostly sufficient to adapt the wave picture. This allows us to use classical field theory based on Maxwell's equations. Of course, in nano-optics, the systems with which the light fields interact are small (single molecules, quantum dots) which necessitates a quantum description of the material properties. Thus, in most cases we can use the framework of semiclassical theory which combines the classical picture of fields and the quantum picture of matter. However, occasionally, we have to go beyond the semiclassical description. For example the photons emitted by a quantum system can obey non-classical photon statistics in form of photon-antibunching.

This section summarizes the fundamentals of electromagnetic theory forming the necessary basis for this course. Only the basic properties are discussed and for more detailed treatments the reader is referred to the books of Chew [27], Jackson [30], Stratton [31], and others. The starting point are Maxwell's equations established by James C. Maxwell in 1873.

1.3.1 Maxwell's equations

In differential form and in *SI* units the *macroscopic* Maxwell's equations read as

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} , \tag{1.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} + \mathbf{j}(\mathbf{r},t) , \tag{1.2}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t) , \tag{1.3}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0 . \tag{1.4}$$

where $\mathbf{E}$ denotes the electric field, $\mathbf{D}$ the electric displacement, $\mathbf{H}$ the magnetic field, $\mathbf{B}$ the magnetic induction, $\mathbf{j}$ the current density, and ρ the charge density. The components of these vector and scalar fields constitute a set of 16 unknowns. Depending on the considered medium, the number of unknowns can be reduced considerably. For example, in linear, isotropic, homogeneous and source free media the electromagnetic field is entirely defined by two scalar fields. Maxwell's equations combine and complete the laws formerly established by Faraday, Ampere, Gauss, Poisson, and others. Since Maxwell's equations are differential equations they do not account for any fields that are constant in space and time. Any such field can therefore be added to the fields. It has to be emphasized that the concept of fields was introduced to explain the transmission of forces from a source to a receiver. The physical observables are therefore forces, whereas the fields are definitions introduced to explain the troublesome phenomenon of the 'action at a distance'. Notice, that the macroscopic Maxwell equations deal with fields which are local spatial averages over microscopic fields associated with discrete charges. Hence, the microscopic nature of matter is not included in the macroscopic fields. Charge and current densities are considered as continuous functions of space. In order to describe the fields on an atomic scale it is necessary to use the microscopic Maxwell equations which consider all matter to be made of charged and uncharged particles.

The conservation of charge is implicitly contained in Maxwell's equations. Taking the divergence of Eq. 1.2, noting that $\nabla \cdot \nabla \times \mathbf{H}$ is identical zero, and substituting Eq. 1.3 for $\nabla \cdot \mathbf{D}$ one obtains the continuity equation

$$\nabla \cdot \mathbf{j}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = 0 . \tag{1.5}$$

The electromagnetic properties of the medium are most commonly discussed in terms of the macroscopic polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ according to

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_o \mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t) , \tag{1.6}$$

$$\mathbf{H}(\mathbf{r},t) = \mu_o^{-1} \mathbf{B}(\mathbf{r},t) - \mathbf{M}(\mathbf{r},t) , \tag{1.7}$$

where ε_o and μ_o are the permittivity and the permeability, respectively. These equations do not impose any conditions on the medium and are therefore always valid.

1.3.2 Wave equations

After substituting the fields $\mathbf{D}$ and $\mathbf{B}$ in Maxwell's *curl* equations by the expressions 1.6 and 1.7 and combining the two resulting equations we obtain the inhomogeneous wave equations

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_o \frac{\partial}{\partial t}\left(\mathbf{j} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}\right) , \tag{1.8}$$

$$\nabla \times \nabla \times \mathbf{H} + \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla \times \mathbf{j} + \nabla \times \frac{\partial \mathbf{P}}{\partial t} + \mu_o \frac{\partial \mathbf{M}}{\partial t} . \tag{1.9}$$

The constant c was introduced for $(\varepsilon_o \mu_o)^{-1/2}$ and is known as the vacuum speed of light. The expression in the brackets of Eq. 1.8 can be associated with the *total current density*

$$\mathbf{j}_t = \mathbf{j}_s + \mathbf{j}_c + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} , \tag{1.10}$$

where $\mathbf{j}$ has been split into a *source current density* $\mathbf{j}_s$ and an induced *conduction current density* $\mathbf{j}_c$. The terms $\partial \mathbf{P}/\partial t$ and $\nabla \times \mathbf{M}$ are recognized as the *polarization current density* and the *magnetization current density*, respectively. The wave equations as stated in Eqs. 1.8, 1.9 do not impose any conditions to the media considered and hence are generally valid.

1.3.3 Constitutive relations

Maxwell's equations define the fields that are generated by currents and charges in matter. However, they do not describe how these currents and charges are generated. Thus, to find a self-consitent solution for the electromagnetic field, Maxwell's equations must be supplemented by relations which describe the behavior of matter under the influence of the fields. These material equations are known as . In a non-dispersive linear and isotropic medium they have the form

$$\mathbf{D} = \varepsilon_o \varepsilon \, \mathbf{E} \qquad (\mathbf{P} = \varepsilon_o \, \chi_e \, \mathbf{E}) \, , \tag{1.11}$$

$$\mathbf{B} = \mu_o \mu \, \mathbf{H} \qquad (\mathbf{M} = \chi_m \, \mathbf{H}) \, , \tag{1.12}$$

$$\mathbf{j}_c = \sigma \mathbf{E} \, . \tag{1.13}$$

For *non-linear* media, the right-hand sides can be supplemented by terms of higher power, but most materials behave linear if the fields are not too strong. *Anisotropic* media can be considered using tensorial forms for ε and μ. In order to account for general *bianisotropic* media, additional terms relating $\mathbf{D}$ and $\mathbf{E}$ to both $\mathbf{B}$ and $\mathbf{H}$ have to be introduced. For such complex media, solutions to the wave equations can be found for very special situations only. The constituent relations given above account for *inhomogeneous* media if the material parameters ε, μ and σ are functions of space. The medium is called *temporally dispersive* if the material parameters are functions of frequency, and *spatially dispersive* if the constitutive relations are convolutions over space. An electromagnetic field in a linear medium can be written as a superposition of monochromatic fields of the form[1]

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{k}, \omega) \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \, , \tag{1.14}$$

where $\mathbf{k}$ and ω are the wavevector and the angular frequency, respectively. In its most general form, the amplitude of the induced displacement $\mathbf{D}(\mathbf{r}, t)$ can be written as

$$\mathbf{D}(\mathbf{k}, \omega) = \varepsilon_o \, \varepsilon(\mathbf{k}, \omega) \; \mathbf{E}(\mathbf{k}, \omega) \, . \tag{1.15}$$

Since $\mathbf{E}(\mathbf{k}, \omega)$ is equivalent to the Fourier transform $\hat{\mathbf{E}}$ of an arbitrary time-dependent field $\mathbf{E}(\mathbf{r}, t)$, we can apply the inverse Fourier transform to Eq. 1.15 and obtain

$$\mathbf{D}(\mathbf{r}, t) = \varepsilon_o \iint \tilde{\varepsilon}(\mathbf{r} - \mathbf{r}', t - t') \; \mathbf{E}(\mathbf{r}', t') \; d\mathbf{r}' \, dt' \, . \tag{1.16}$$

Here, $\tilde{\varepsilon}$ denotes the inverse Fourier transform (spatial and temporal) of ε. The above equation is a convolution in space and time. The displacement $\mathbf{D}$ at time t depends on the electric field at all times t' previous to t (temporal dispersion). Additionally, the displacement at a point $\mathbf{r}$ also depends on the values of the electric field at neighboring points $\mathbf{r}'$ (spatial dispersion). A spatially dispersive medium is therefore also called a *nonlocal* medium. Nonlocal effects can be observed at interfaces between different media or in metallic objects with sizes comparable with the mean-free path of electrons. In general, it is very difficult to account for spatial dispersion in field calculations. In most cases of interest the effect is very week and we can savely ignore it. Temporal dispersion, on the other hand, is a widely encountered phenomenon and it is important to accurately take it into account.

[1] In an anisotropic medium the dielectric constant $\varepsilon = \overleftrightarrow{\varepsilon}$ is a second-rank tensor.

1.3.4 Arbitrary time-dependent fields

The spectrum $\hat{\mathbf{E}}(\mathbf{r},\omega)$ of an arbitrary time-dependent field $\mathbf{E}(\mathbf{r},t)$ is defined by the Fourier transform

$$\hat{\mathbf{E}}(\mathbf{r},\omega) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)\, \mathrm{e}^{i\omega t}\, dt \;. \tag{1.17}$$

In order that $\mathbf{E}(\mathbf{r},t)$ is a real valued field we have to require that

$$\hat{\mathbf{E}}(\mathbf{r},-\omega) \;=\; \hat{\mathbf{E}}^{*}(\mathbf{r},\omega) \;. \tag{1.18}$$

Applying the Fourier transform to the time-dependent Maxwell equations (Eqs. 1.1 - 1.4) gives

$$\nabla\times\hat{\mathbf{E}}(\mathbf{r},\omega) \;=\; i\omega\,\hat{\mathbf{B}}(\mathbf{r},\omega)\;, \tag{1.19}$$
$$\nabla\times\hat{\mathbf{H}}(\mathbf{r},\omega) \;=\; -i\omega\,\hat{\mathbf{D}}(\mathbf{r},\omega) \,+\, \hat{\mathbf{j}}(\mathbf{r},\omega)\;, \tag{1.20}$$
$$\nabla\cdot\hat{\mathbf{D}}(\mathbf{r},\omega) \;=\; \hat{\rho}(\mathbf{r},\omega)\;, \tag{1.21}$$
$$\nabla\cdot\hat{\mathbf{B}}(\mathbf{r},\omega) \;=\; 0\;, \tag{1.22}$$

Once the solution for $\hat{\mathbf{E}}(\mathbf{r},\omega)$ has been determined, the time-dependent field is calculated by the inverse transform as

$$\mathbf{E}(\mathbf{r},t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{E}}(\mathbf{r},\omega)\, \mathrm{e}^{-i\omega t}\, d\omega \;. \tag{1.23}$$

Thus, the time-dependence of a non-harmonic electromagnetic field can be Fourier transformed and every spectral component can be treated separately as a monochromatic field. The general time dependence is obtained from the inverse transform.

1.3.5 Time harmonic fields

The time dependence in the wave equations can be easily separated to obtain a harmonic differential equation. A monochromatic field can then be written as[2]

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\{\mathbf{E}(\mathbf{r})\,\mathrm{e}^{-i\omega t}\} \;=\; \frac{1}{2}\left[\mathbf{E}(\mathbf{r})\,\mathrm{e}^{-i\omega t} +\; \mathbf{E}^{*}(\mathbf{r})\,\mathrm{e}^{i\omega t}\right]\;, \tag{1.24}$$

with similar expressions for the other fields. Notice that $\mathbf{E}(\mathbf{r},t)$ is real, whereas the spatial part $\mathbf{E}(\mathbf{r})$ is complex. The symbol $\mathbf{E}$ will be used for both, the real, time-dependent field and the complex spatial part of the field. The introduction of a new symbol is avoided in order to keep the notation simple. It is convenient to represent the fields of a time-harmonic field by their complex amplitudes. Maxwell's equations can then be written as

$$\nabla\times\mathbf{E}(\mathbf{r}) \;=\; i\omega\,\mathbf{B}(\mathbf{r})\;, \tag{1.25}$$
$$\nabla\times\mathbf{H}(\mathbf{r}) \;=\; -i\omega\,\mathbf{D}(\mathbf{r}) \,+\, \mathbf{j}(\mathbf{r})\;, \tag{1.26}$$
$$\nabla\cdot\mathbf{D}(\mathbf{r}) \;=\; \rho(\mathbf{r})\;, \tag{1.27}$$
$$\nabla\cdot\mathbf{B}(\mathbf{r}) \;=\; 0\;, \tag{1.28}$$

[2] This can also be written as $\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\{\mathbf{E}(\mathbf{r})\}\cos\omega t + \mathrm{Im}\{\mathbf{E}(\mathbf{r})\}\sin\omega t \;=\; |\mathbf{E}(\mathbf{r})|\cos[\omega t + \varphi(\mathbf{r})]$, where the phase is determined by $\varphi(\mathbf{r}) = \arctan[\mathrm{Im}\{\mathbf{E}(\mathbf{r})\}/\mathrm{Re}\{\mathbf{E}(\mathbf{r})\}]$

which is equivalent to Maxwell's equations 1.19 - 1.22 for the spectra of arbitrary time-dependent fields. Thus, the solution for $\mathbf{E}(\mathbf{r})$ is equivalent to the spectrum $\hat{\mathbf{E}}(\mathbf{r},\omega)$ of an arbitrary time-dependent field. It is obvious that the complex field amplitudes depend on the angular frequency ω, i.e. $\mathbf{E}(\mathbf{r}) = \mathbf{E}(\mathbf{r},\omega)$. However, ω is usually not included in the argument. Also the material parameters ε, μ, and σ are functions of space and frequency, i.e. $\varepsilon=\varepsilon(\mathbf{r},\omega)$, $\sigma=\sigma(\mathbf{r},\omega)$, $\mu=\mu(\mathbf{r},\omega)$. For simpler notation, we will often drop the argument in the fields and material parameters. It is the context of the problem which determines which of the fields $\mathbf{E}(\mathbf{r},t)$, $\mathbf{E}(\mathbf{r})$, or $\hat{\mathbf{E}}(\mathbf{r},\omega)$ is being considered.

1.3.6 Complex dielectric constant

With the help of the linear constitutive relations we can express Maxwell's curl equations (Eqs. 1.25 and 1.26) in terms of $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$. We then multiply both sides of the first equation by μ^{-1} and then apply the curl operator to both sides. After the expression $\nabla \times \mathbf{H}$ is substituted by the second equation we obtain

$$\nabla \times \mu^{-1}\,\nabla \times \mathbf{E} \;-\; \frac{\omega^2}{c^2}\,[\varepsilon + i\sigma/(\omega\varepsilon_o)]\;\mathbf{E} \;=\; i\omega\mu_o\mathbf{j}_s\;. \qquad (1.29)$$

It is common practice to replace the expression in the brackets on the left hand side by a complex dielectric constant, i.e.

$$[\varepsilon \,+\, i\,\sigma/(\omega\,\varepsilon_o)] \;\rightarrow\; \varepsilon\;. \qquad (1.30)$$

In this notation one does not distinguish between conduction currents and polarization currents. Energy dissipation is associated with the imaginary part of the dielectric constant. With the new definition of ε, the wave equations for the complex fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ in linear, isotropic, but *inhomogeneous* media are

$$\nabla \times \mu^{-1}\,\nabla \times \mathbf{E} \;-\; k_o^2\,\varepsilon\,\mathbf{E} \;=\; i\omega\mu_o\mathbf{j}_s\;, \qquad (1.31)$$

$$\nabla \times \varepsilon^{-1}\,\nabla \times \mathbf{H} \;-\; k_o^2\,\mu\,\mathbf{H} \;=\; \nabla \times\; \varepsilon^{-1}\,\mathbf{j}_s\;, \qquad (1.32)$$

where $k_o = \omega/c$ denotes the vacuum wave number. These equations are also valid for anisotropic media if the substitutions $\varepsilon \rightarrow \overleftrightarrow{\varepsilon}$ and $\mu \rightarrow \overleftrightarrow{\mu}$ are performed. The complex dielectric constant will be used throughout this book.

1.3.7 Piecewise homogneous media

In many physical situations the medium is piecewise homogeneous. In this case the entire space is divided into subdomains in which the material parameters are independent of position $\mathbf{r}$. In principle, a piecewise homogeneous medium is inhomogeneous and the solution can be derived from Eqs. 1.31, 1.32. However, the inhomogeneities are entirely confined to the boundaries and it is convenient to formulate the solution for each subdomain separately. These solutions must be connected with each other via the interfaces to form the solution for all space. Let the interface between to homogeneous domains D_i and D_j be denoted as

∂D_{ij}. If ε_i and μ_i designate the constant material parameters in subdomain D_i, the wave equations in that domain read as

$$(\nabla^2 + k_i^2)\,\mathbf{E}_i \;=\; -i\omega\,\mu_o\mu_i\,\mathbf{j}_i + \frac{\nabla\rho_i}{\varepsilon_o\,\varepsilon_i}\;, \tag{1.33}$$

$$(\nabla^2 + k_i^2)\,\mathbf{H}_i \;=\; -\nabla\times\mathbf{j}_i\;, \tag{1.34}$$

where $k_i = (\omega/c)\sqrt{\mu_i\,\varepsilon_i}$ is the wavenumber and $\mathbf{j}_i$, ρ_i the sources in domain D_i. To obtain these equations, the identity $\nabla\times\nabla\times = -\nabla^2 + \nabla\nabla\cdot$ was used and Maxwell's equation 1.3 was applied. Eqs. 1.33 and 1.34 are also denoted as the inhomogeneous vector Helmholtz equations. In most practical applications, such as scattering problems, there are no source currents or charges present and the Helmholtz equations are homogeneous.

1.3.8 Boundary conditions

Since the material properties are discontinuous on the boundaries, Eqs. 1.33 and 1.34 are only valid in the interior of the subdomains. However, Maxwell's equations must also hold for the boundaries. Due to the discontinuity it turns out to be difficult to apply the differential forms of Maxwell's equations but there is no problem with the corresponding integral forms. The latter can be derived by applying the theorems of Gauss and Stokes to the differential forms 1.1 - 1.4 and read as

$$\int_{\partial S}\mathbf{E}(\mathbf{r},t)\cdot d\mathbf{s} \;=\; -\int_S \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}_s\;da\;, \tag{1.35}$$

$$\int_{\partial S}\mathbf{H}(\mathbf{r},t)\cdot d\mathbf{s} \;=\; \int_S\left[\mathbf{j}(\mathbf{r},t) + \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right]\cdot\mathbf{n}_s\;da\;, \tag{1.36}$$

$$\int_{\partial V}\mathbf{D}(\mathbf{r},t)\cdot\mathbf{n}_s\;da \;=\; \int_V \rho(\mathbf{r},t)\;dV\;, \tag{1.37}$$

$$\int_{\partial V}\mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}_s\;da \;=\; 0\,. \tag{1.38}$$

In these equations, da denotes a surface element, $\mathbf{n}_s$ the normal unit vector to the surface, $d\mathbf{s}$ a line element, ∂V the surface of the volume V, and ∂S the border of the surface S. The integral forms of Maxwell equations lead to the desired boundary conditions if they are applied to a sufficiently small part of the considered boundary. In this case the boundary looks flat and the fields are homogeneous on both sides [Fig. 1.4]. Consider a small rectangular path ∂S along the the boundary as shown in Fig 1.4a. As the area S (enclosed by the path ∂S) is arbitrarily reduced, the electric and magnetic flux through S become zero. This does not necessarily apply for the source current, since a surface current density $\mathbf{K}$ might be present. The first two Maxwell equations then lead to the boundary conditions for the tangential field components[3]

$$\mathbf{n}\times(\mathbf{E}_i - \mathbf{E}_j) \;=\; \mathbf{0}\quad on\ \partial D_{ij}\;, \tag{1.39}$$

$$\mathbf{n}\times(\mathbf{H}_i - \mathbf{H}_j) \;=\; \mathbf{K}\quad on\ \partial D_{ij}\;, \tag{1.40}$$

[3]Notice, that $\mathbf{n}$ and $\mathbf{n}_s$ are different unit vectors: $\mathbf{n}_s$ is perpendicular to the surfaces S and ∂V, whereas $\mathbf{n}$ is perpendicular to the boundary ∂D_{ij}.

where $\mathbf{n}$ is the unit normal vector on the boundary. A relation for the normal components can be obtained by considering an infinitesimal rectangular box with volume V and surface ∂V according to Fig. 1.4b. If the fields are considered to be homogeneous on both sides and if a surface charge density σ is assumed, Maxwell's third and fourth equation lead to the boundary conditions for the normal field components

$$\mathbf{n} \cdot (\mathbf{D}_i - \mathbf{D}_j) = \sigma \quad \textit{on } \partial D_{ij} \tag{1.41}$$

$$\mathbf{n} \cdot (\mathbf{B}_i - \mathbf{B}_j) = 0 \quad \textit{on } \partial D_{ij} \,. \tag{1.42}$$

In most practical situations there are no sources in the individual domains, and $\mathbf{K}$ and σ consequently vanish. The four boundary conditions 1.39 - 1.41 are not independent from each other since the fields on both sides of ∂D_{ij} are linked by Maxwell's equations. It can be easily shown, for example, that the conditions for the normal components are automatically satisfied if the boundary conditions for the tangential components hold everywhere on the boundary and Maxwell's equations are fulfilled in both domains.

1.3.9 Conservation of energy

The equations established so far describe the behavior of electric and magnetic fields. They are a direct consequence of Maxwell's equations and the properties of matter. Although the electric and magnetic fields were initially postulated to explain the forces in Coulomb's and Ampere's laws, Maxwell's equations do not provide any information about the energy or forces in a system. The basic Lorentz' law discribes the forces acting on moving charges only. As the Abraham-Minkowski controversy shows, the forces acting on an arbitrary object cannot be extracted from a given electrodynamic field in a consistent way. It is also interesting, that Coulomb's and Ampere's laws were sufficient to establish Lorentz' force law. While later the field equations have been completed by adding the Maxwell displacement current,

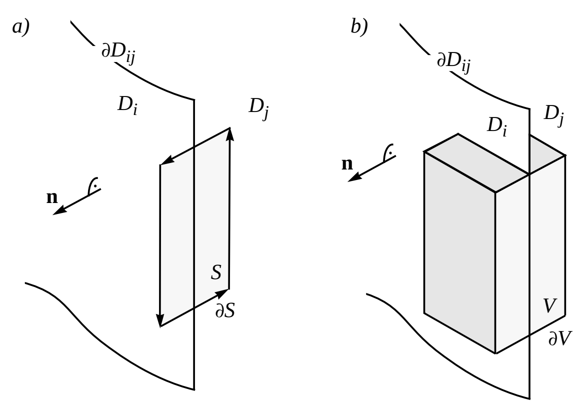

Figure 1.4: Integration paths for the derivation of the boundary conditions on the interface ∂D_{ij} between two adjacent domains D_i and D_j.

the Lorentz law remained unchanged. There is less controversy regarding the energy. Although also not a direct consequence of Maxwell's equations, Poynting's theorem provides a plausible relationship between the electromagnetic field and its energy content. For later reference, Poynting's theorem shall be outlined below.

If the scalar product of the field $\mathbf{E}$ and Eq. 1.2 is subtracted from the scalar product of the field $\mathbf{H}$ and Eq. 1.1 the following equation is obtained

$$\mathbf{H}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H}) \;=\; -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \mathbf{j}\cdot\mathbf{E}\,. \tag{1.43}$$

Noting that the expression on the left is identical to $\nabla\cdot(\mathbf{E}\times\mathbf{H})$, integrating both sides over space and applying Gauss' theorem the equation above becomes

$$\int_{\partial V}(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n}\,da \;=\; -\int_V\Big[\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{j}\cdot\mathbf{E}\Big]\,dV\,. \tag{1.44}$$

Although this equation already forms the basis of Poynting's theorem, more insight is provided when $\mathbf{B}$ and $\mathbf{D}$ are substituted by the generally valid equations 1.6 and 1.7. Eq. 1.44 then reads

$$\int_{\partial V}(\mathbf{E}\times\mathbf{H})\cdot\mathbf{n}\,da \;+\; \frac{1}{2}\frac{\partial}{\partial t}\int_V\Big[\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}\Big]\,dV \;=\; \tag{1.45}$$

$$-\int_V\mathbf{j}\cdot\mathbf{E}\,dV \;-\; \frac{1}{2}\int_V\Big[\mathbf{E}\cdot\frac{\partial\mathbf{P}}{\partial t} - \mathbf{P}\cdot\frac{\partial\mathbf{E}}{\partial t}\Big]\,dV \;-\; \frac{\mu_o}{2}\int_V\Big[\mathbf{H}\cdot\frac{\partial\mathbf{M}}{\partial t} - \mathbf{M}\cdot\frac{\partial\mathbf{H}}{\partial t}\Big]\,dV\,.$$

This equation is a direct conclusion of Maxwell's equations and has therefore the same validity. Poynting's theorem is more or less an interpretation of the equation above. It states that the first term is equal to the net energy flow in or out of the volume V, the second term is equal to the time rate of change of electromagnetic energy inside V and the remaining terms on the right side are equal to the rate of energy dissipation inside V. According to this interpretation

$$\mathbf{S} \;=\; (\mathbf{E}\times\mathbf{H}) \tag{1.46}$$

represents the energy flux density and

$$W \;=\; \frac{1}{2}\Big[\mathbf{D}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{H}\Big] \tag{1.47}$$

is the density of electromagnetic energy. If the medium within V is linear, the two last terms equal zero and the only term accounting for energy dissipation is $\mathbf{j}\cdot\mathbf{E}$. Hence, the two last terms can be associated with non-linear losses. The vector $\mathbf{S}$ is denoted as the Poynting vector. In principle, the curl of any vector field can be added to $\mathbf{S}$ without changing the conservation law 1.45, but it is convenient to make the choice as stated in 1.46. Notice that the current $\mathbf{j}$ in Eq. 1.45 is the current associated with energy dissipation and therefore does not include polarization and magnetization currents.

Of special interest is the mean time value of $\mathbf{S}$. This quantity describes the net power flux density and is needed for the evaluation of radiation patterns. Assuming that the fields are harmonic in time and that the media are linear, the time average of Eq. 1.45 becomes

$$\int_{\partial V}\langle\mathbf{S}\rangle\cdot\mathbf{n}\,da \;=\; -\frac{1}{2}\int_V\mathrm{Re}\,\{\mathbf{j}^*\cdot\mathbf{E}\}\,dV, \tag{1.48}$$

where the term on the right defines the mean energy dissipation within the volume V. $\langle \mathbf{S} \rangle$ represents the time average of the Poynting vector

$$\langle \mathbf{S} \rangle \;=\; \frac{1}{2}\,\mathrm{Re}\left\{\mathbf{E}\times\mathbf{H}^*\right\} \;. \tag{1.49}$$

In the far-field, the electromagnetic field is purely transverse. Furthermore, the electric and magnetic fields are in phase and the ratio of their amplitudes is constant. In this case $\langle \mathbf{S} \rangle$ can be expressed by the electric field alone as

$$\langle \mathbf{S} \rangle \;=\; \frac{1}{2}\sqrt{\frac{\varepsilon_o\,\varepsilon}{\mu_o\,\mu}}\;|\mathbf{E}|^2\,\mathbf{n}_r \;, \tag{1.50}$$

where $\mathbf{n}_r$ represents the unit vector in radial direction and the inverse of the square root denotes the wave impedance.