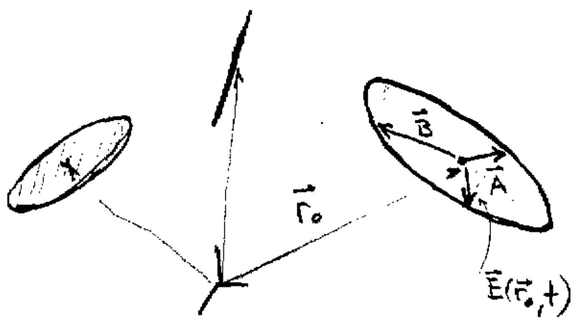
POLARIZATION AND CRYSTAL OPTICS

(A) What is the polarization state of an unspecified monochromatic field ?

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\vec{E}(\vec{r})\, e^{-i\omega t}\right\}$$

$$= \mathrm{Re}\left\{\vec{E}(\vec{r})\right\}\cos\omega t + \mathrm{Im}\left\{\vec{E}(\vec{r})\right\}\sin\omega t$$

For a fixed position $\vec{r} = \vec{r}_0$ in space: $\vec{E}(\vec{r}_0,t) = \underbrace{\vec{A}\cos\omega t + \vec{B}\sin\omega t}_{\text{ellipse!}}$

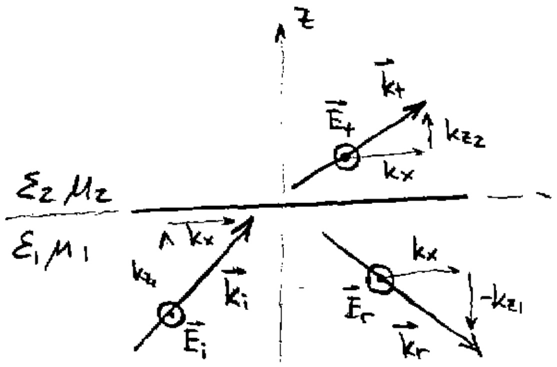


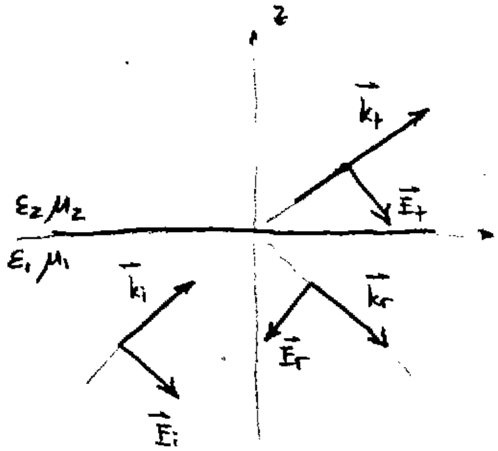
The endpoint of the vector $\vec{E}(\vec{r}_0,t)$ draws an ellipse with center $\vec{r}_0$.

In general, each point in space has different axes $\vec{A}$ and $\vec{B}$. The ellipses are different from point to point. In special cases they form circles or lines.

Polarized light has the property that all ellipses in space have the same shape. The size of an ellipse is characteristic for the field strength in a particular point. Therefore, the size distribution is not necessarily uniform.

B) Polarizing properties of interfaces





s-polarized incidence:

$$\frac{\vec{E}_r \cdot \vec{n}_{E_r}}{\vec{E}_i \cdot \vec{n}_{E_i}} = r^s = \frac{\mu_2 k_{z_1} - \mu_1 k_{z2}}{\mu_2 k_{z_1} + \mu_1 k_{z2}}$$

$$\frac{\vec{E}_t \cdot \vec{n}_{E_t}}{\vec{E}_i \cdot \vec{n}_{E_i}} = t^s = \frac{2\mu_2 k_{z_1}}{\mu_2 k_{z_1} + \mu_1 k_{z2}}$$

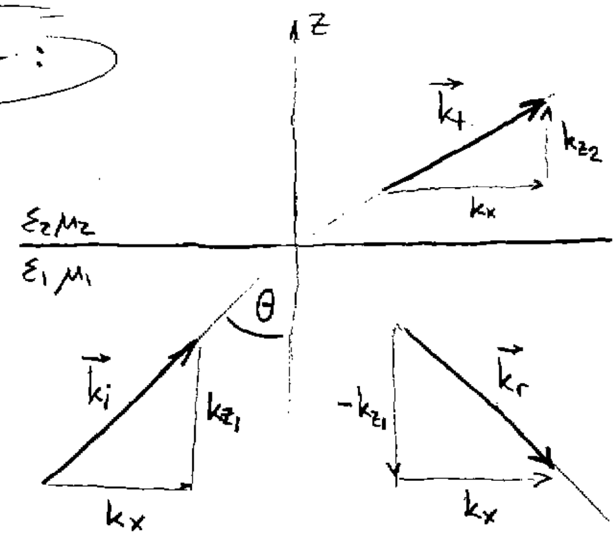
p-polarized incidence:

$$\frac{\vec{E}_r \cdot \vec{n}_{E_r}}{\vec{E}_i \cdot \vec{n}_{E_i}} = r^p = \frac{\varepsilon_2 k_{z_1} - \varepsilon_1 k_{z2}}{\varepsilon_2 k_{z_1} + \varepsilon_1 k_{z2}} \qquad (1)$$

$$\frac{\vec{E}_t \cdot \vec{n}_{E_t}}{\vec{E}_i \cdot \vec{n}_{E_i}} = t^p = \sqrt{\frac{\mu_2 \varepsilon_1}{\mu_1 \varepsilon_2}}\, \frac{2\varepsilon_2 k_{z1}}{\varepsilon_2 k_{z1} + \varepsilon_1 k_{z2}} \qquad (2)$$

Fresnel reflection/transmission coefficients are generally valid if expressed in terms of k_{z_1}, k_{z2}. This is not the case if they are expressed in terms of incoming and outgoing angles!

Important:



$$|\vec{k}_i|^2 = |\vec{k}_r|^2 = \frac{\omega^2}{c^2}\mu_1 \varepsilon_1 = k_1^2$$

$$|k_t|^2 = \frac{\omega^2}{c^2}\mu_2 \varepsilon_2 = k_2^2$$

$$k_{x_i} = k_{x_r} = k_{x_t} = k_x$$

$$\rightarrow \sqrt{k_1^2 - k_{z_1}^2} = \sqrt{k_2^2 - k_{z_2}^2}$$

$$\boxed{\text{for } k_{z_1} = k_1 \cos\theta : \; k_{z2} = k_2 \sqrt{1 - \frac{k_1^2}{k_2^2}\sin^2\theta}} \qquad (3)$$

× careful: for ideal conductor $\varepsilon_2 \to -\infty$ and $k_{z_2} = 0$:

$$\Longrightarrow \quad r^s = -1 \quad \text{but} \quad r^p = 1 \quad ! \tag{4}$$

× Optical frequencies: $\mu_1 = \mu_2 \approx 1$

× Brewster angle: Angle for which all light is transmitted, i.e. for which the reflection coefficient is zero.

s-polarization: $r^s = 0$ if $k_{z_1} = k_{z_2}$

→ not possible if the two materials are different.

p-polarization: $r^p = 0$ if $\varepsilon_1 k_{z_2} = \varepsilon_2 k_{z_1}$

→ use Eq. 3: $\varepsilon_2 k_1 \cos\theta_B = \varepsilon_1 k_2 \sqrt{1 - \frac{k_1^2}{k_2^2}\sin^2\theta_B}$

$$\to \quad \boxed{\sin\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}} \tag{5}$$

or using $\frac{\sin^2\theta}{1-\sin^2\theta} = \tan^2\theta$:

$$\to \quad \boxed{\tan\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}} \tag{6}$$

× critical angle of total internal reflection: $k_{z_2} = 0$

Using Eq. (3): $1 - \frac{k_1^2}{k_2^2}\sin^2\theta_c = 0$

or: $$\boxed{\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}} \qquad (7)$$

This can only happen if $\varepsilon_2 < \varepsilon_1$, i.e. the wave is incident from the optically denser medium!

<u>$\theta > \theta_c$:</u>

Total internal reflection (k_{z_2} = imaginary).

<u>$\theta < \theta_c$:</u>

Refraction (k_{z_2} = real)

Important: A certain $\vec{k}$-vector is associated with a plane wave or an evanescent wave. Any field can be written as superpositions of plane waves and evanescent waves (angular spectrum representation).

Consider plane wave: $$\vec{E}_i(\vec{r},t) = \mathrm{Re}\{\underbrace{\vec{E}_i\, e^{i\vec{k}_i\cdot\vec{r}}}_{=\vec{E}_i(\vec{r})}\, e^{-i\omega t}\}$$

The transmitted wave is: $$\vec{E}_t(\vec{r},t) = \mathrm{Re}\{\vec{E}_t\, e^{i\vec{k}_t\cdot\vec{r}}\, e^{-i\omega t}\}$$

$$= \mathrm{Re}\{\vec{E}_t\, e^{ik_x x + ik_y y}\, e^{ik_{z_2}z}\, e^{-i\omega t}\}$$

$e^{ik_{z_2}z}$ = oscillating for $\theta < \theta_c$

$e^{ik_{z_2}z}$ = exponentially decaying for $\theta > \theta_c$

(c) Wave propagation in anisotropic crystals

Maxwell equations for monochromatic fields ($\vec{B} = \mu_o \vec{H}$):

$$\begin{aligned} \nabla \cdot \vec{D} &= 0 \\ \nabla \cdot \vec{H} &= 0 \\ \nabla \times \vec{E} &= i\omega\mu_o \vec{H} \\ \nabla \times \vec{H} &= -i\omega \vec{D} \end{aligned} \tag{8}$$

Combine curl equations:

$$\nabla \times \nabla \times \vec{E} = i\omega\mu_o (\nabla \times \vec{H}) = \omega^2 \mu_o \vec{D} \tag{9}$$

Consider a plane wave:

$$\begin{aligned} \vec{E} &= \vec{E}_o e^{i\vec{k}\cdot\vec{r}} \\ \vec{D} &= \vec{D}_o e^{i\vec{k}\cdot\vec{r}} \end{aligned} \tag{10}$$

$$\begin{aligned} \longrightarrow \quad \nabla \times \vec{E} &= i\vec{k} \times \vec{E} \\ \nabla \cdot \vec{E} &= i\vec{k} \cdot \vec{E} \end{aligned} \tag{11}$$

Using Eq. (11) in Eq. (9): $\nabla \times \nabla \times \vec{E} = -\vec{k} \times \vec{k} \times \vec{E}$

$$= -\vec{k}(\vec{k}\cdot\vec{E}) + \vec{E}(\vec{k}\cdot\vec{k})$$

$$\longrightarrow \quad \vec{D} = \frac{-1}{\mu_o \omega^2} \left[-\vec{k}(\vec{k}\cdot\vec{E}) + \vec{E} k^2 \right]$$

Set $k = \frac{\omega}{c} n$, where $n = \sqrt{\varepsilon}$ = effective index of refraction :

$$\boxed{\vec{D} = n^2 \varepsilon_0 \left[\vec{E} - \frac{1}{k^2}\vec{k}(\vec{k}\cdot\vec{E})\right]} \qquad (12)$$

Write $\vec{D} = \varepsilon_0 \overleftrightarrow{\varepsilon}\vec{E}$ with $\overleftrightarrow{\varepsilon}$ being a diagonalized 3×3 matrix :

$$\varepsilon_0 \overleftrightarrow{\varepsilon}\vec{E} = n^2 \varepsilon_0 \left[\vec{E} - \frac{1}{k^2}\vec{k}(\vec{k}\cdot\vec{E})\right]$$

$$n_i^2 E_i = n^2 E_i - \frac{n}{k^2} k_i (\vec{k}\cdot\vec{E}) \quad \text{with } i = x,y,z \quad \text{and} \quad n_i^2 = \varepsilon_i$$

$$(n^2 - n_i^2) E_i = \frac{n}{k^2} k_i (\vec{k}\cdot\vec{E})$$

$$E_i = \frac{k_i n}{k^2(n^2 - n_i^2)} (\vec{k}\cdot\vec{E}) \quad \longrightarrow \quad k_i E_i = \frac{k_i^2 n}{k^2(n^2-n_i^2)} (\vec{k}\cdot\vec{E}) \qquad (13)$$

Take sum over all i :

$$\underbrace{\sum_i k_i E_i}_{(\vec{k}\cdot\vec{E})} = \left[\sum_i \frac{k_i^2 n}{k^2(n^2-n_i^2)}\right] (\vec{k}\cdot\vec{E})$$

Write out components :

$$\boxed{\frac{1}{n^2} = \frac{k_x^2/k^2}{n^2 - n_x^2} + \frac{k_y^2/k^2}{n^2 - n_y^2} + \frac{k_z^2/k^2}{n^2 - n_z^2}} \qquad (14)$$

Fresnel's Equation

Notice: x, y, z are not arbitrary! They are given by the optical axes of the crystal as defined by the diagonalized $\overleftrightarrow{\varepsilon}$ tensor.

With Eq. (14) we can calculate the effective index n_e for a particular direction of propagation $\vec{k}/k$.

$$(n^2 - n_x^2)(n^2 - n_y^2)(n^2 - n_z^2) = (k_x/k)^2 n^2 (n^2 - n_y^2)(n^2 - n_z^2) + (k_y/k)^2 n^2 (n^2 - n_x^2)(n^2 - n_z^2) + (k_z/k)^2 n^2 (n^2 - n_x^2)(n^2 - n_y^2)$$

$\longrightarrow$ solve for n^2 : n^6 terms cancel, i.e. highest terms are n^4

Therefore $\rightarrow$ 2 solutions for n^2!

2 solutions for n_e imply 2 solutions for $\vec{E}$ field:

$$\boxed{\begin{gathered}\vec{E}(\vec{r}) = \vec{E}_0^{(1)} e^{i\vec{k}^{(1)}\cdot\vec{r}} + \vec{E}_0^{(2)} e^{i\vec{k}^{(2)}\cdot\vec{r}} \\ \text{where } k^{(1)} = \frac{\omega}{c} n^{(1)}, \quad k^{(2)} = \frac{\omega}{c} n^{(2)}\end{gathered}} \tag{15}$$

Important: $\vec{E}_0^{(1)}$ is not necessarily parallel to $\vec{D}_0^{(1)}$, same for $\vec{E}_0^{(2)}, \vec{D}_0^{(2)}$
$\rightarrow$ see Eq. (12)

However $\vec{D}_0^{(1)}$ and $\vec{D}_0^{(2)}$ are perpendicular to $\vec{k}^{(1)}$ and $\vec{k}^{(2)}$, respectively. From Eq. (12): $\vec{k}\cdot\vec{D} = n^2 \varepsilon_0 \left[\vec{k}\cdot\vec{E} - \frac{\vec{k}\cdot\vec{k}}{k^2}(\vec{k}\cdot\vec{E})\right] = 0$

Decompose $\vec{E}$ into 2 components, one parallel to $\vec{k}$, the other perpendicular to $\vec{k}$:

$$\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp} \tag{16}$$

Insert into Eq. (12):

$$\vec{D} = n^2 \varepsilon_0 \Big[\vec{E}_{\parallel} + \vec{E}_{\perp} - \underbrace{\frac{\vec{k}}{k^2}(\vec{k}\cdot\vec{E}_{\parallel}}_{\vec{E}_{\parallel}} + \underbrace{\vec{k}\cdot\vec{E}_{\perp}}_{=0})\Big] = n^2 \varepsilon_0 \vec{E}_{\perp} \tag{17}$$

$\longrightarrow$ $\vec{D}$ is only parallel to one component of $\vec{E}$!

$$\text{Thus: } \boxed{\vec{k}\cdot\vec{D} = 0\,, \quad \vec{k}\cdot\vec{E} \neq 0} \tag{18}$$

<u>Poynting vector:</u>

$$\vec{S} = \vec{E} \times \vec{H} \tag{19}$$

Using $\vec{H} = (i\omega\mu_0)^{-1}\nabla\times\vec{E}$ and $\vec{k} = n\frac{\omega}{c}\cdot\frac{\vec{k}}{k} = n\frac{\omega}{c}\vec{n}_k$ and $\vec{E} = E\,\vec{n}_E$:

$$\vec{S} = \frac{n}{\mu_0 c} E^2 \underbrace{\left[\vec{n}_E \times \vec{n}_k \times \vec{n}_E\right]}_{\vec{n}_k(\vec{n}_E\cdot\vec{n}_E) - \vec{n}_E(\vec{n}_E\cdot\vec{n}_k)} = \frac{n}{\mu_0 c} E^2 \left[\vec{n}_k - \vec{n}_E(\vec{n}_E\cdot\vec{n}_k)\right]$$

$$\vec{n}_E\cdot\vec{S} = \frac{n}{\mu_0 c} E^2 \left[\vec{n}_E\cdot\vec{n}_k - (\vec{n}_E\cdot\vec{n}_E)(\vec{n}_E\cdot\vec{n}_k)\right] = 0$$

$$\longrightarrow \boxed{\vec{S} \text{ is } \perp \text{ to } \vec{E}} \tag{20}$$

Decompose $\vec{n}_k$ into component parallel to $\vec{S}$ and component perpendicular to $\vec{S}$:

$$\vec{n}_K = \vec{n}_{K\parallel} + \vec{n}_{K\perp} \tag{21}$$

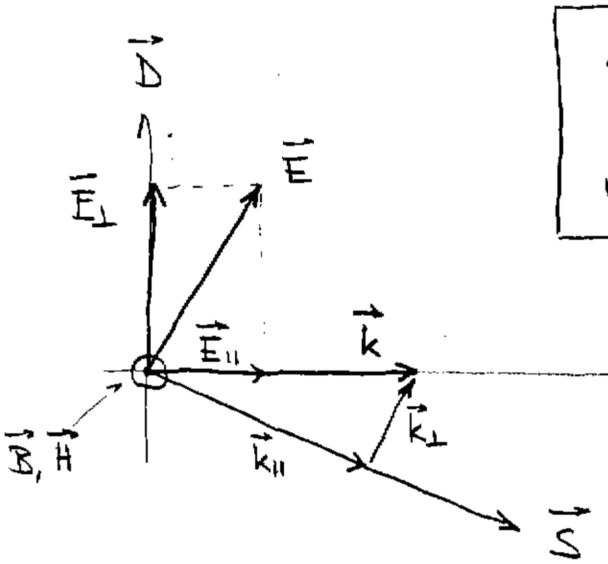
$$\vec{S} = \frac{n}{\mu_0 c} E^2 \left[\vec{n}_{K\parallel} + \vec{n}_{K\perp} - \vec{n}_E \left(\underbrace{\vec{n}_E \cdot \vec{n}_{K\parallel}}_{0} + \underbrace{\vec{n}_E \cdot \vec{n}_{K\perp}}_{1} \right) \right]$$

$$= \frac{n}{\mu_0 c} E^2 \left[\vec{n}_{K\parallel} + \underbrace{(\vec{n}_{K\perp} - \vec{n}_E)}_{=0} \right]$$

Thus: $\vec{S} = \frac{n}{\mu_0 c} E^2 \, \vec{n}_{K\parallel}$

$\rightarrow$ $\boxed{\text{Phase velocity } \vec{V}_p = \frac{c}{n} \vec{n}_K \text{ is not } \parallel \text{ to } \vec{S}}$

Sketch:

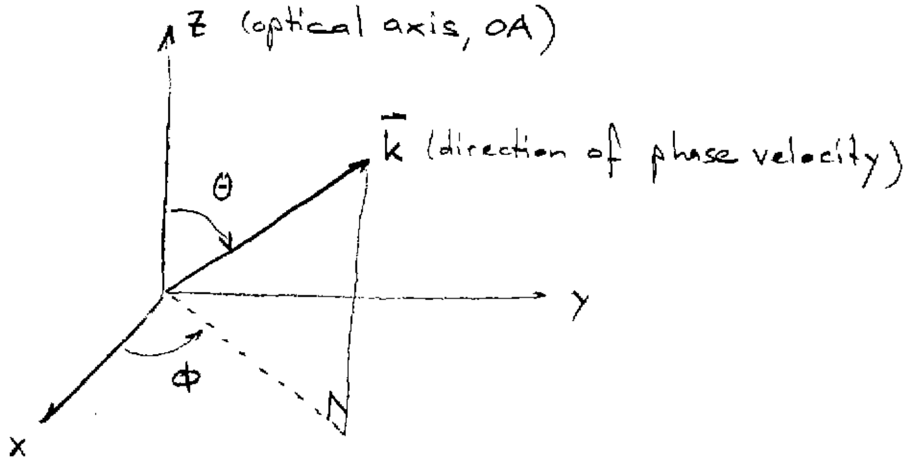


$\boxed{\text{Energy is not transported in direction of the wavefront normal !}}$

Uniaxial crystals (birefringence)

$n_x = n_y = n_o$

$n_z = n_e$



Index 'o' refers to ordinary wave ($\vec{E} \perp OA$)

Index 'e' refers to extraordinary wave ($\vec{E}$ not $\perp OA$)

Rotations in ϕ do not change the physical situation ($n_x = n_y$)

$\longrightarrow$ consider $\vec{k}$ in the x,z plane ($k_y = 0$)

Consider Eq. 12 together with $\vec{D}_i = \varepsilon_0 n_i^2 \vec{E}_i$ and $k_y = 0$:

$$n_o^2 E_x = n^2 \left[E_x - \frac{1}{k^2} k_x (k_x E_x + k_z E_z) \right] \tag{22}$$

$$n_o^2 E_y = n^2 E_y \tag{23}$$

$$n_e^2 E_z = n^2 \left[E_z - \frac{1}{k^2} k_z (k_x E_x + k_z E_z) \right] \tag{24}$$

Ordinary wave ($\vec{E} \perp OA$) : Eq. 23 $\longrightarrow$ $\boxed{n = n_o}$

Extraordinary wave ($\vec{E}$ not $\perp \vec{OA}$) : Eq. 22 and 23 $\longrightarrow$ $\boxed{n = \dfrac{n_e n_o}{\sqrt{n_e^2 (k_z/k)^2 + n_o^2 (k_x/k)^2}}}$ (25)

determinant = 0

Rewrite Eq. (25):

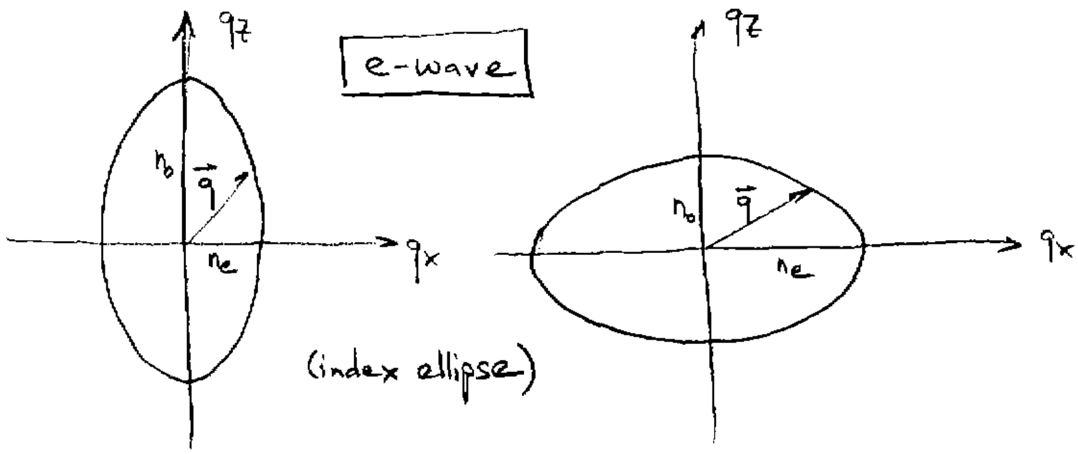
$$\boxed{\frac{1}{n^2} = \frac{k_x^2/k^2}{n_e^2} + \frac{k_z^2/k^2}{n_o^2}} \qquad \text{compare with Eq. 14} \tag{26}$$

Extraordinary wave

Define: $q_x = n\, k_x/k = n \sin\theta$

$q_z = n\, k_z/k = n \cos\theta$

Eq. (26) becomes
$$\frac{q_x^2}{n_e^2} + \frac{q_z^2}{n_z^2} = 1 \quad \text{(index ellipse)} \tag{27}$$



$n_e < n_o$ (calcite, $CaCO_3$) (negative uniaxial Xtal)

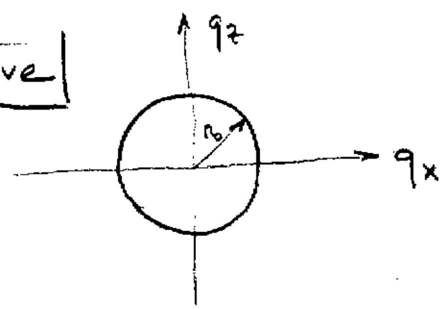
$n_e > n_o$ (rutile, TiO_2) (positive uniaxial Xtal)

Ordinary wave

o-wave

$$q = \sqrt{q_x^2 + q_z^2} = n = n_o \tag{28}$$

(sphere)

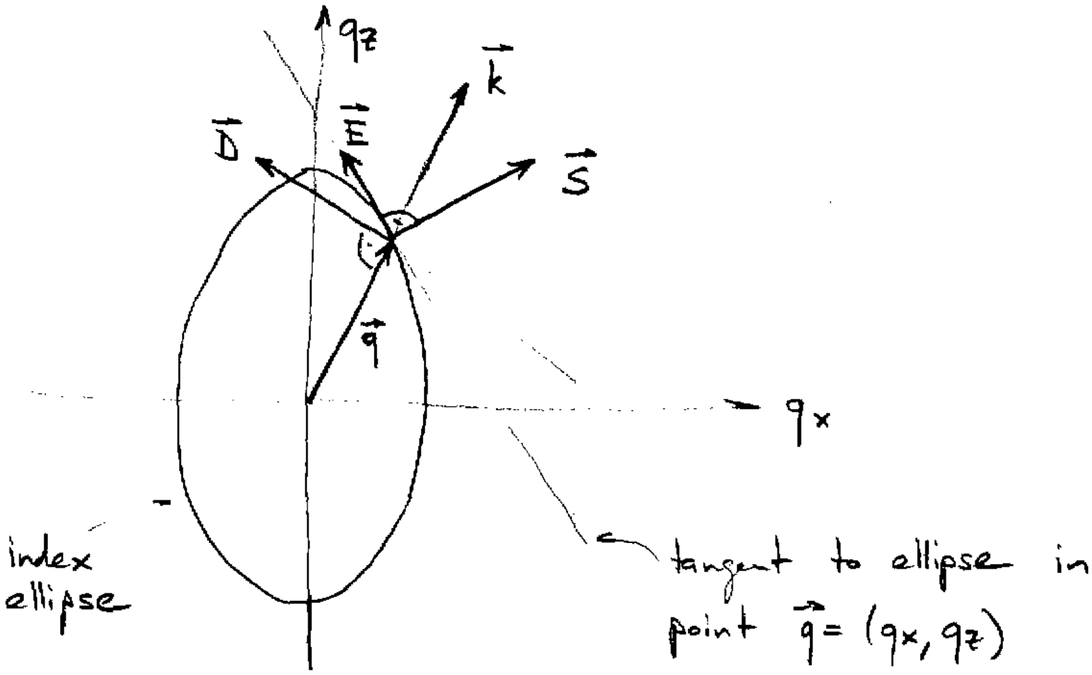


Determine $\vec{E}$-field of extraordinary wave

From Eq. (22) : $$E_z / E_x = \frac{n^2 k_z^2/k^2 - n_o^2}{n^2 k_x k_z / k^2} \qquad (k^2 = k_x^2 + k_y^2) \tag{29}$$

From Eq. (26) : $$n^2 = \frac{n_e^2\, n_o^2}{n_o^2\, k_x^2/k^2 + n_e^2\, k_z^2/k^2} \tag{30}$$

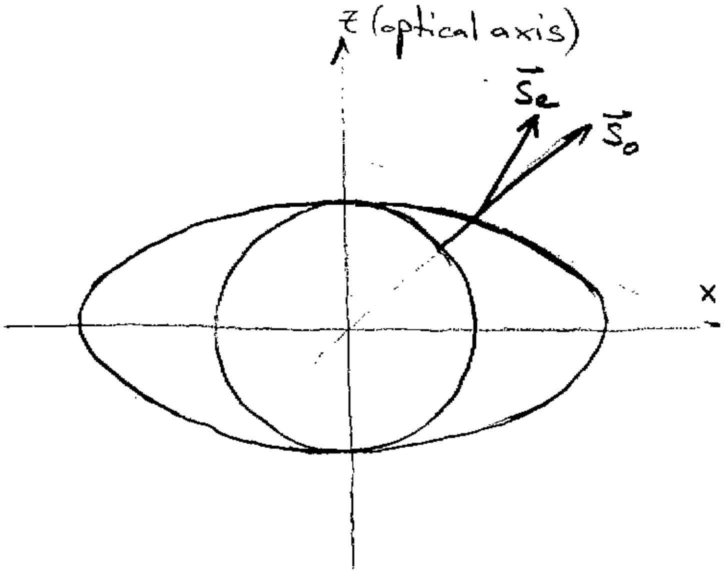
Combine : $$\boxed{\frac{E_z}{E_x} = -\frac{n_o^2}{n_e^2}\frac{k_x}{k_z} = -\frac{n_o^2}{n_e^2}\frac{q_x}{q_z}} \tag{31}$$



$$\boxed{\begin{array}{l}\vec{E} \text{ is in direction of the tangent in point } \vec{q} \\ \vec{S} \text{ is } \perp \text{ on } \vec{E}\end{array}} \tag{32}$$

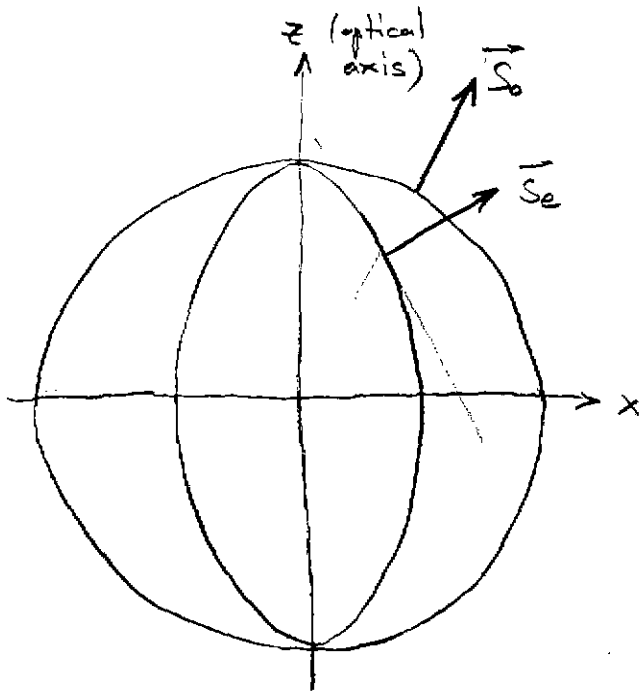
e-wave and o-wave are identical if $k_x = 0$. In this case $k_z = k$ and Eq. (26) yields $n = n_o$, which is identical to Eq. (28).

Two cases depending on whether $n_o > n_e$ (positive uniaxial crystal) or $n_o < n_e$ (negative uniaxial crystal).



$n_e > n_o$

negative uniaxial crystal



$n_e < n_o$

positive uniaxial crystal

Examples:

① Waveplates: Propagation $\perp$ to optical axis: $\vec{S}_e \parallel \vec{S}_o$

If $\vec{E} \perp$ to optical axis $\rightarrow$ only o-wave

If $\vec{E} \parallel$ to optical axis $\rightarrow$ only e-wave

Phase velocities: $v_p = \begin{cases} c/n_o & \text{(ordinary wave)} \\ c/n_e & \text{(extraordinary wave)} \end{cases}$

$$\vec{E} = \vec{E}_z e^{ik_1 x} + \vec{E}_y e^{ik_2 x} = e^{ik_1 x}\left[\vec{E}_z + \vec{E}_y \underbrace{e^{i(k_2-k_1)x}}_{e^{\pm i\Delta\varphi}}\right]$$

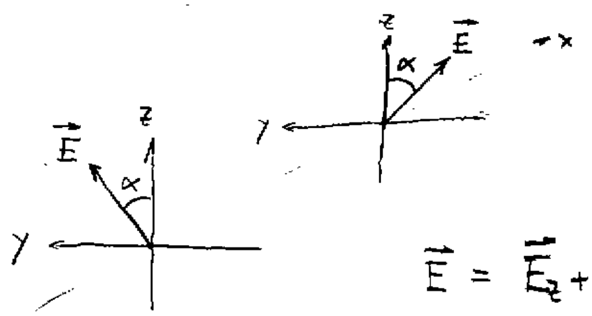
phase difference: $\boxed{\Delta\varphi = \frac{2\pi}{\lambda}|n_o - n_e|\, x}$ (33)

(1.1) full-wave plate: $\Delta\varphi = 2\pi \qquad \left(x = N \frac{\lambda}{|n_o - n_e|}\right)$ (34)

e-wave and o-wave are back in phase

(1.2) half-wave plate: $\Delta\varphi = \pi \qquad \left(x = \frac{2N+1}{2} \frac{\lambda}{|n_o - n_e|}\right)$ (35)

e-wave and o-wave are 180° out of phase



axis with lower index is called the <u>fast axis</u>.

$$\vec{E} = \vec{E}_z + \vec{E}_y \qquad \text{input}$$

$$\vec{E} = e^{ik_1 x}\left[\vec{E}_z - \vec{E}_y\right] \qquad \text{output} \tag{36}$$

set: $\left.\begin{aligned}\vec{E}_y &= E \sin\alpha\, \vec{n}_y \\ \vec{E}_z &= E \cos\alpha\, \vec{n}_z\end{aligned}\right\}$ linearly polarized light

$$\longrightarrow \vec{E} = E e^{ik_1 x}\left[\cos\alpha\, \vec{n}_z - \sin\alpha\, \vec{n}_y\right] \tag{37}$$

(flips polarization)

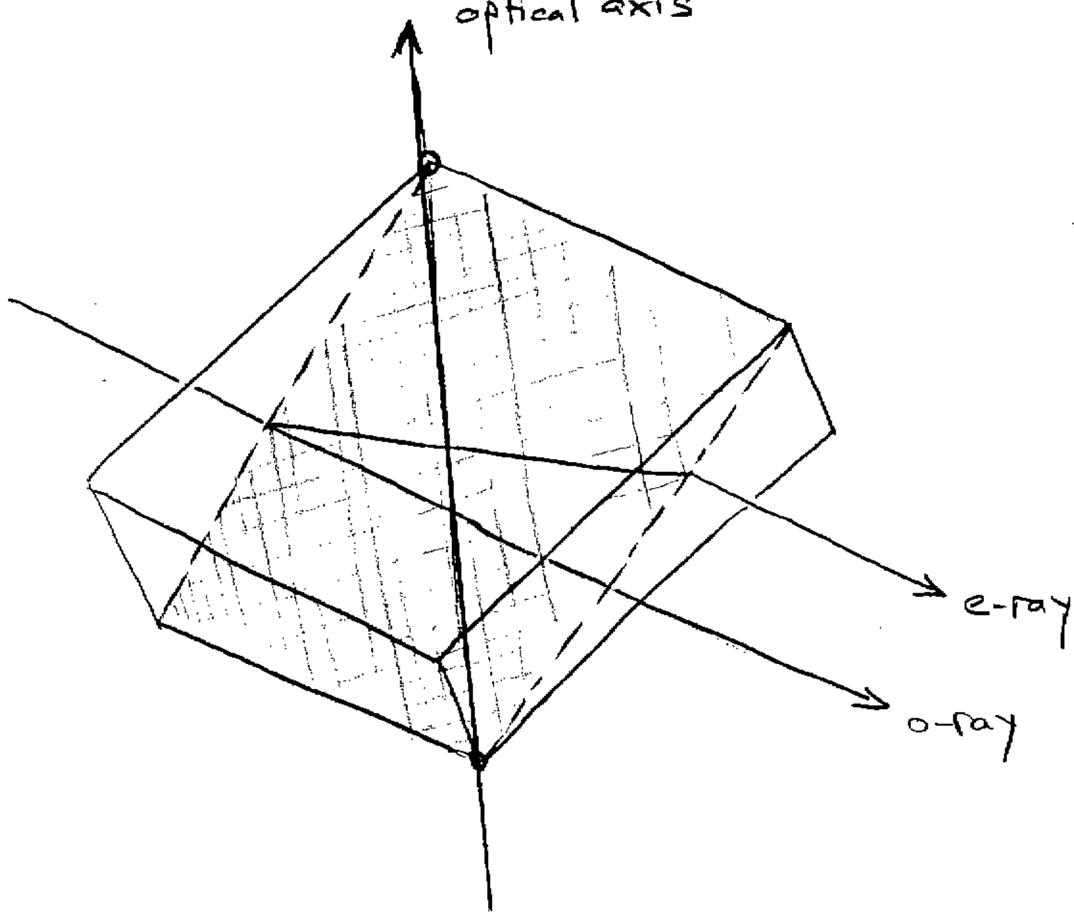
$\alpha = 0, \pi/2 \rightarrow$ no change in polarization

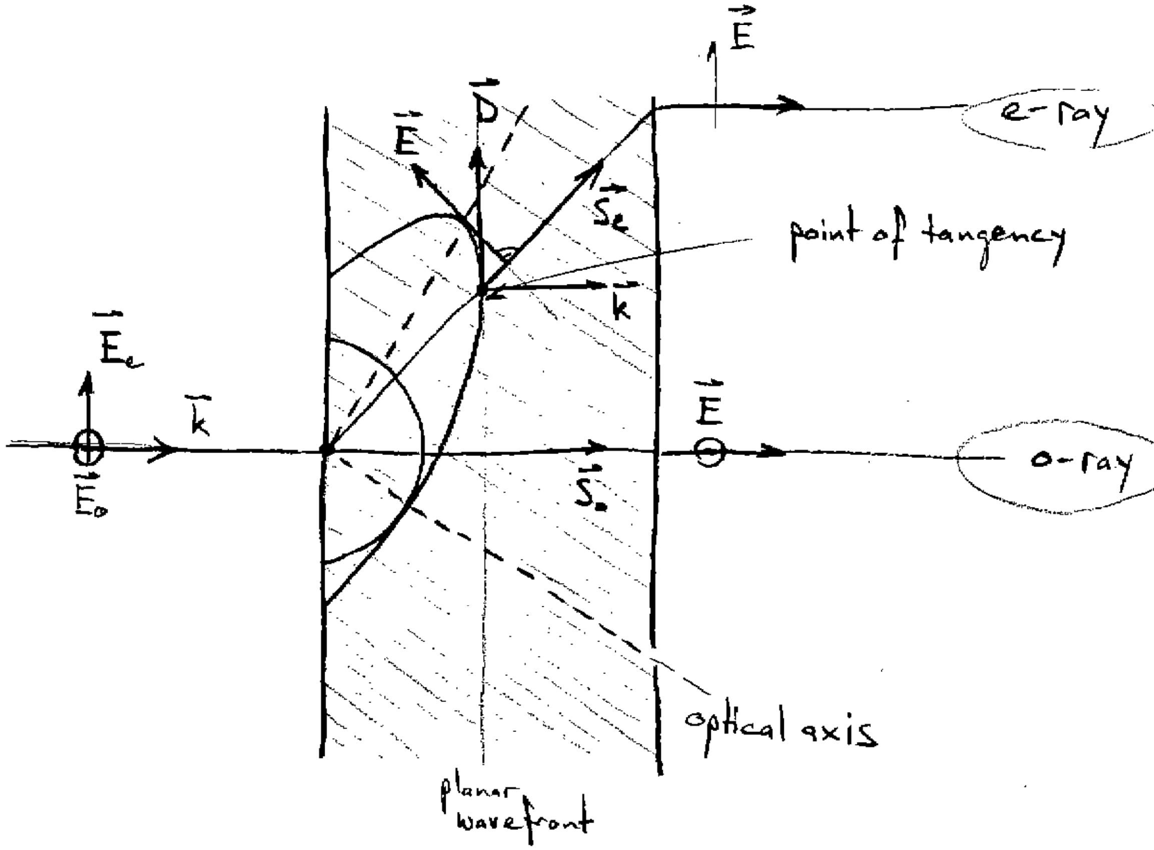
(1.3) quarter-wave plate: $\Delta\varphi = \pi/2 \qquad \left(x = \frac{4N+1}{4} \frac{\lambda}{|n_o - n_e|}\right)$ (38)

consider linearly polarized input: $\vec{E} = E e^{ik_1 x}\left[\cos\alpha\, \vec{n}_z + i \sin\alpha\, \vec{n}_y\right]$ (39)

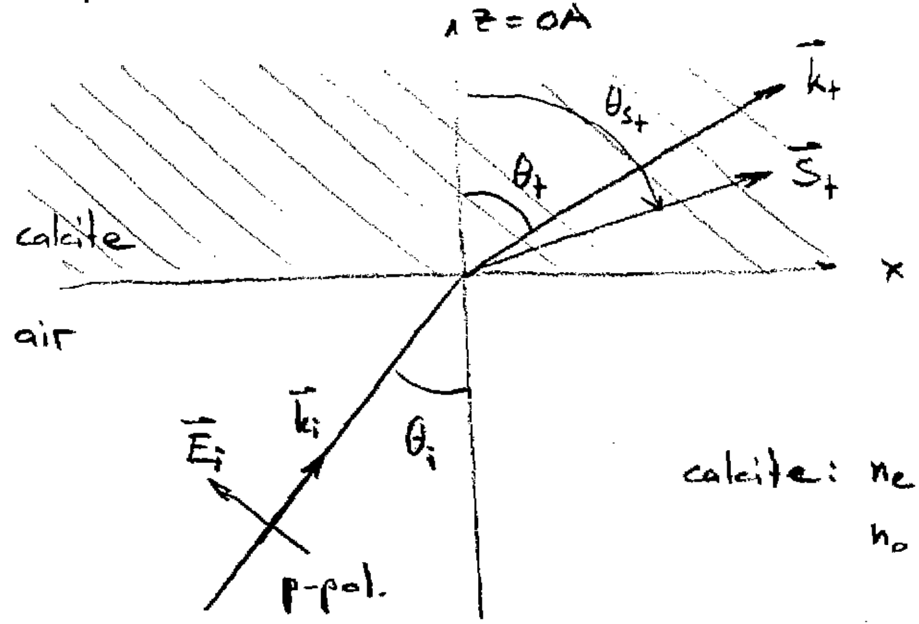
$\left.\begin{aligned}&\hookrightarrow \alpha = 0, \pi/2 && \text{no change} \\ &\alpha = \pi/4 && \text{circularly polarized output} \\ &\alpha \neq 0, \pi/4, \pi/2 && \text{elliptically polarized output}\end{aligned}\right\}$ works also in reverse!

② Birefringence :





Calculated example:



calcite: $n_e = 1.486$

$n_o = 1.658$

(1) $k_{x_i} = k_{x_t}$: $\quad k_i \sin\theta_i = k_t \sin\theta_t \longrightarrow \sin\theta_i = n\sin\theta_t$

(Snell's law)

(2) $\dfrac{q_z^2}{n_o^2} + \dfrac{q_x^2}{n_e^2} = 1$: $\quad q_z = n_o\sqrt{1-(q_x/n_e)^2}$

where $q_x = n\,k_x/k$, $q_z = n\,k_z/k$

(1+2) $$\tan\theta_t = \frac{q_x}{q_z} = \frac{q_x}{n_o\sqrt{1-(q_x/n_e)^2}} = \frac{n\,k_x/k}{n_o\sqrt{1-(n\,k_x/k\,/n_e)^2}}$$

$$= \frac{n\sin\theta_t}{n_o\sqrt{1-(n\sin\theta_t/n_e)^2}} = \frac{\sin\theta_i}{n_o\sqrt{1-(\sin\theta_i/n_e)^2}} \Longrightarrow \underline{\underline{\theta_t}}$$

(3) $\dfrac{E_z}{E_x} = -\dfrac{n_o^2}{n_e^2}\dfrac{q_x}{q_z} \longrightarrow \vec{S}_t \perp$ on $\vec{E}$: $\quad \dfrac{S_x}{S_z} = -\dfrac{E_z}{E_x}$

$$\longrightarrow \tan\theta_{s_t} = \frac{S_x}{S_z} = \frac{n_o^2}{n_e^2}\tan\theta_t \Longrightarrow \underline{\underline{\theta_{s_t}}}$$