# Kramers-Kronig relations

The Kramers-Kronig relations imply that the real and imaginary parts of materialparameters, such as $\varepsilon(\omega)$, $\chi(\omega)$, or $\kappa(\omega)$, are <u>not independent</u> but are connected by integral relations. The Kramers-Kronig relations are unexpected on physical grounds. They are purely a consequence of <u>linearity</u> and <u>causality</u>.

Consider a linear functional relationship between two time-dependent functions $\vec{P}(t)$ and $\vec{E}(t)$:

$$\vec{P}(t) = \varepsilon_o \int_{-\infty}^{\infty} \tilde{\chi}(t,t')\, \vec{E}(t')\, dt' \qquad (1)$$

For $\vec{E}(t') = \vec{E}_o\, \delta(t-t')$ : $\vec{P}(t) = \varepsilon_o \tilde{\chi}(t,t_o)\vec{E}_o$

If the properties of the medium do not change with time, $\vec{P}$ must only depend on the the time elapsed between $t_o$ and $t$

$$\longrightarrow \tilde{\chi}(t,t_o) = \tilde{\chi}(t-t_o)$$

Causality implies that the system ($\vec{P}$) cannot recpond before it has been excited ($\vec{E}$). Thus: $\tilde{\chi}(t-t_o) = 0$ for $t < t_o$.

Fourier transform: $$\chi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\chi}(\tau) e^{i\omega\tau} d\tau = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \tilde{\chi}(\tau) e^{i\omega\tau} d\tau \qquad (2)$$

where $\tau = t - t_o$

Generalize $\omega$ as complex variable $s$ with $\mathrm{Im}\{s\} \geq 0$:

$$\chi(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \tilde{\chi}(\tau)\, e^{is\tau}\, d\tau \qquad \text{(Laplace transform)} \tag{3}$$

On the real axis of $s$, $\chi(s)$ coincides with $\chi(\omega)$.
$\tilde{\chi}(\tau) e^{is\tau}$ is an analytic function of $s$. Because of $\mathrm{Im}\{s\} \geq 0$, we have for times $\tau \geq 0$: $|\tilde{\chi}(\tau) e^{is\tau}| \leq |\tilde{\chi}(\tau)|$.
Therefore, if the integral

$$\int_0^\infty |\tilde{\chi}(\tau)|\, d\tau \tag{4}$$

converges, then the integral in Eq. (3) also converges in the upper half-space of the s-plane. Thus, provided Eq. (4) is true, <u>$\chi(s)$ is an analytic function</u>. If $\chi(s)$ is an analytic function then also

$$f(s) = \frac{\chi(s)}{(s-\omega)}$$

is an analytic function except at the pole $s = \omega$ which is chosen on the real axis of $s$. Cauchy's theorem states that

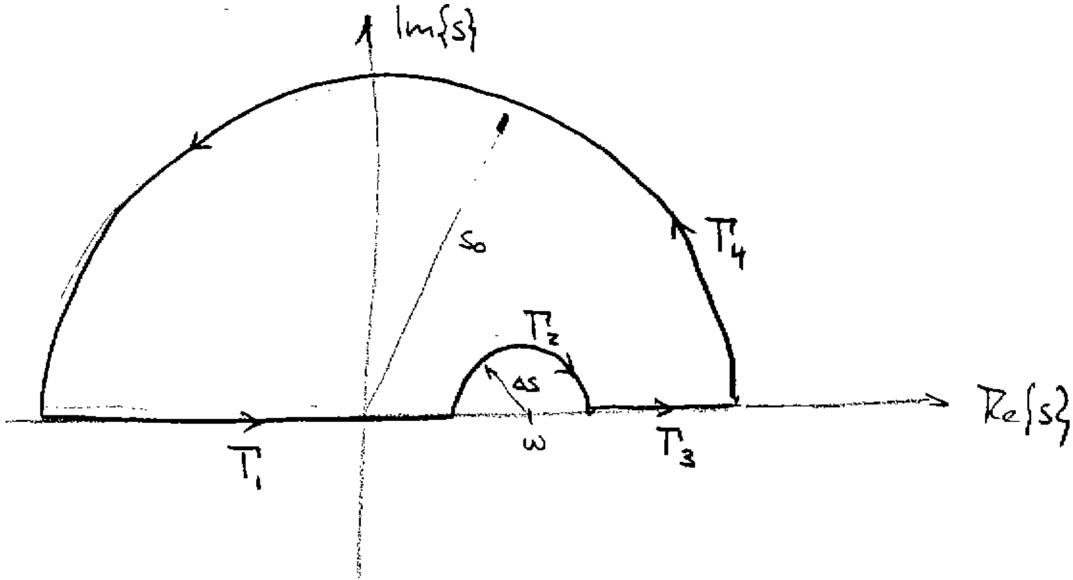
$$\oint_\Gamma f(s)\, ds = 0 \tag{5}$$

provided that the closed path $\Gamma$ does not enclose any poles.

Let us evaluate the integral

$$\oint_{\Gamma} \frac{(s)}{(s-\omega)}\, ds \tag{6}$$

on the contour shown below




The contour $\Gamma$ is divided into four curves. Obviously, the integral vanishes on $\Gamma_4$ if we let $s_0 \to \infty$. Therefore,

$$\int_{-\infty}^{\omega-\Delta s} \frac{\chi(s)}{(s-\omega)}\, ds + \int_{\omega+\Delta s}^{\infty} \frac{\chi(s)}{(s-\omega)}\, ds = \underbrace{-\int_0^{\pi} \frac{\chi(\omega-\Delta s\, e^{-i\varphi})}{[\omega-\Delta s\, e^{-i\varphi}]-\omega}\,(i\Delta s\, e^{-i\varphi})\, d\varphi}_{i\int_0^{\pi} \chi(\omega-\Delta s\, e^{-i\varphi})\, d\varphi}$$

Let $\Delta s \to 0$;

$$\lim_{\Delta s\to 0}\left[\int_{-\infty}^{\omega-\Delta s} \frac{\chi(s)}{(s-\omega)}\, ds + \int_{\omega+\Delta s}^{\infty} \frac{\chi(s)}{(s-\omega)}\, ds\right] = i\pi\,\chi(\omega) \tag{7}$$

or, with P denoting the Cauchy principal value:

$$\boxed{\chi(\omega) = \frac{1}{i\pi} P\left[\int_{-\infty}^{\infty} \frac{\chi(s)}{s-\omega}\, ds\right]} \tag{8}$$

Making use of $\chi^*(s) = \chi(-s)$, Eq. (8) can be written as

$$\chi(\omega) = \frac{1}{i\pi} P\left[\int_0^{\infty} \frac{\chi(s)}{s-\omega}\, ds - \int_0^{\infty} \frac{\chi(s)}{s+\omega}\, ds\right] . \tag{9}$$

Taking the real and imaginary part of Eq. (9) gives:

$$\boxed{\begin{aligned} \chi'(\omega) &= \frac{2}{\pi} P\left[\int_0^{\infty} \frac{s\,\chi''(s)}{s^2-\omega^2}\, ds\right] \\ \chi''(\omega) &= -\frac{2\omega}{\pi} P\left[\int_0^{\infty} \frac{\chi'(s)}{s^2-\omega^2}\, ds\right] \end{aligned}} \tag{10}$$

Thus, if the real part of $\chi(\omega)$ is known for all $\omega$, the imaginary part of $\chi(\omega)$ is uniquely determined (and vice versa) !

Notice that

$$\int_0^\infty |\tilde{\varepsilon}(\tau)|\, d\tau$$

does not converge. Therefore, the Kramers-Kronig relations as derived in Eq. (10) do not hold for $\varepsilon(\omega)$. Instead, we have to use $\varepsilon(\omega) = 1 + \chi(\omega)$ and obtain

$$\boxed{\begin{aligned} \varepsilon'(\omega) &= 1 + \frac{2}{\pi} P\left[\int_0^\infty \frac{s\ \varepsilon''(s)}{s^2-\omega^2}\, ds\right] \\ \varepsilon''(\omega) &= -\frac{2\omega}{\pi} P\left[\int_0^\infty \frac{\varepsilon'(s)-1}{s^2-\omega^2}\, ds\right] \end{aligned}} \qquad (11)$$

Are there materials with no absorption? No! If $\varepsilon''(\omega) = 0$ then $\varepsilon'(\omega) = 1$ which is the case for vacuum.