What is an electromagnetic field?

Fields $\vec{E}, \vec{B}$ were introduced in order to overcome the dilemma of 'action at distance'.

moving a charged object at one position x_1 $\longrightarrow$ causes another charged object at a different position x_2 to respond

nothing in between! $\longrightarrow$ NOTHING?

Measurable physical quantities = Forces

Coulomb force

static charges attract or repel

$\downarrow$

introduce electric field to mediate force from x_1 to x_2:

$$\vec{F}_e = q\vec{E}$$

ELECTROSTATICS

Lorentz force

static currents (charges with const. velocities) attract or repel

$\downarrow$

introduce magnetic field to mediate force from x_1 to x_2:

$$\vec{F}_m = q\vec{v} \times \vec{B}$$

MAGNETOSTATICS

Thus: The real physical quantity $\vec{F}$ is connected to $\vec{E}, \vec{B}$ through

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}} . \qquad (1)$$

$\vec{E}, \vec{B}$ are mathematical functions introduced to avoid the idea of 'action at distance'.

The question now is whether $\vec{F}$ is real?! The force is the most important quantity in classical physics (Napoleon). Later, it was realized that the potential of applying force (energy E) is even more important (industrialization).

Energy and momentum ($\vec{p}$) are the relevant quantities in quantum mechanics. Surprisingly, it is not $\vec{E}$ and $\vec{B}$ which connect electrodynamics with quantum mechanics but the vector and scalar potentials ($\vec{A}, \phi$).

$$\text{Classical Physics } (\vec{F}) \longleftarrow \vec{E}, \vec{B}$$

$$\text{Quantum Physics } (E, \vec{p}) \longleftarrow \vec{A}, \phi$$

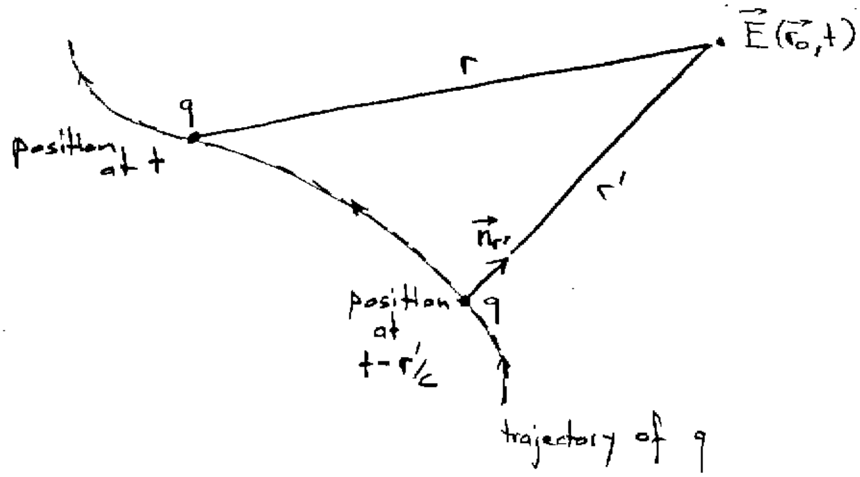
Example: Aharanov-Bohm effect. The path (wavefunction) of a particle is influenced by $\vec{A}$ but not by $\vec{B}$ (regions of zero $\vec{B}$ but nonzero $\vec{A}$).

Relativity: $\vec{E}, \vec{B}$ not independent! $\vec{E}$ in one inertial frame is equal to $\vec{B}$ in another inertial frame

observer at 'rest' sees magnetic field

observer moving at same speed sees electric field.

Electric field at a fixed position $\vec{r}_0$ measured at time t generated by an arbitrary charge q:



$$\boxed{\vec{E}(\vec{r}_0,t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{n}_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{\vec{n}_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\vec{n}_{r'}\right]} \qquad (2)$$

R. Feynman 'Lectures on Physics', Vol. II, p 21-1

The field at $\vec{r}_0$ at time t depends on position and motion of charge at the earlier time $(t - r'/c)$! The time difference $\Delta t = r'/c$ is needed for the electric field to travel the distance r'.

1st term: proportional to <u>position</u> of charge (retarded Coulomb field)

2nd term: proportional to <u>velocity</u> of charge (correction to retarded Coulomb field). This term compensates retardation.
1st + 2nd term together are the instantaneous Coulomb field at the time t.

3rd term: proportional to <u>acceleration</u> (electromagnetic radiation)

Mathematics:

1) Gauss's theorem (divergence theorem)

$$\oint_{\partial V} \vec{F} \cdot \vec{n} \, da = \int_V \nabla \cdot \vec{F} \, dV$$

volume element

surface element

closed surface enclosing V

normal vector

vector field

2) Stokes's theorem

$$\oint_{\partial A} \vec{F} \cdot \vec{ds} = \int_A (\nabla \times \vec{F}) \cdot \vec{n} \, da$$

line element

closed curve bounding A

3) $$\nabla \equiv \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} , \quad \nabla \cdot \vec{F} \equiv \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z$$

$$\nabla \times \vec{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \partial/\partial y \, F_z - \partial/\partial z \, F_y \\ \partial/\partial z \, F_x - \partial/\partial x \, F_z \\ \partial/\partial x \, F_y - \partial/\partial y \, F_x \end{bmatrix} , \quad \nabla^2 \phi = \frac{\partial^2}{\partial x^2} \phi + \frac{\partial^2}{\partial y^2} \phi + \frac{\partial^2}{\partial z^2} \phi$$

4) identities
$$\nabla \times \nabla \phi = 0$$
$$\nabla \cdot (\nabla \times \vec{F}) = 0$$
$$\nabla \times (\nabla \times \vec{F}) = \nabla (\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

see Jackson front and back cover

Pre-Maxwell Electromagnetism (before 1865)

(a) $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ — Gauss's law (Cavendish 1772) (Coulomb 1785)

(b) $\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$ — Faraday's law (Faraday 1825)

(c) $\nabla \times \vec{B} = \mu_0 \vec{j}$ — Ampère's law (Oersted 1819) (Biot, Savard 1820) (Ampère 1820-1825)

(d) $\nabla \cdot \vec{B} = 0$ — No magnetic monopoles

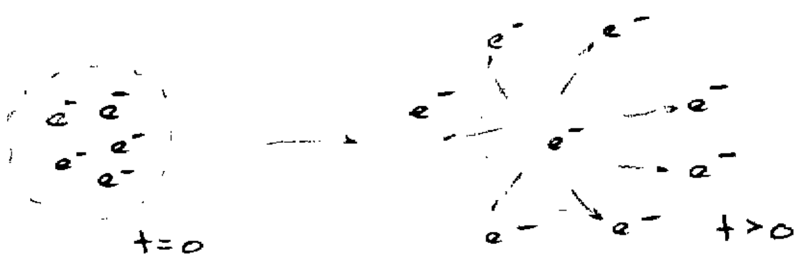
Electrostatics: $\nabla \cdot \vec{E} = \rho/\varepsilon_0$, $\nabla \times \vec{E} = 0$ $\longrightarrow$ $\vec{E} = -\nabla\phi$ $\rightarrow$ $\nabla^2\phi = -\rho/\varepsilon_0$

Magnetostatics: $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{B} = \mu_0\vec{j}$ $\longrightarrow$ $\vec{B} = \nabla \times \vec{A}$ $\rightarrow$ $\nabla^2\vec{A} = -\mu_0\vec{j}$
($\nabla \cdot \vec{A} = 0$)

Since $\nabla \cdot (\nabla \times \vec{B}) = 0$, Eq. (c) gives

$$\nabla \cdot \vec{j} = 0 \quad \text{always!}$$

This is obviously wrong. Take bunch of identical charges and hold them together. Once released, the charges will spread out because of Coulombic repulsion. There will be a net outward current



thus $\oint_{S} \vec{j} \cdot \vec{n}\, da \neq 0 \longrightarrow \nabla \cdot \vec{j} \neq 0$!!

Correction :

$$\boxed{\nabla\cdot\vec{j} = -\frac{\partial}{\partial t}\rho} \quad \text{Conservation of charge} \qquad (3)$$

outward current is balanced by decrease of charge inside V.

This form of equation is encountered in various physical situations. It is called a <u>continuity equation</u>.

We need to modify Ampère's law so that Eq. (3) is satisfied.

J.C. Maxwell 1865 : $\nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$

$\hookrightarrow$ displacement current

$$\underbrace{\nabla\cdot\nabla\times\vec{B}}_{=0} = \mu_0\,\nabla\cdot\vec{j} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\underbrace{\nabla\cdot\vec{E}}_{=\rho/\varepsilon_0}$$

$$\text{thus: } \nabla\cdot\vec{j} + \frac{\partial}{\partial t}\rho = 0 \quad \text{q.e.d.}$$

Equations (a) .. (d) with corrected Ampère's law are called <u>Maxwell's equations</u>. They form the basis of all classical electrodynamic phenomena. For example, electromagnetic radiation

(b) $\nabla\times\nabla\times\vec{E} = -\frac{\partial}{\partial t}\nabla\times\vec{B}$ $\qquad$ with $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

$\hookrightarrow = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$

$\rightarrow \nabla(\rho/\varepsilon_0) - \nabla^2\vec{E}$

$$\boxed{\left[\nabla^2 - \frac{\partial^2}{\partial t^2}\frac{1}{c^2}\right]\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\nabla\rho(\vec{r},t) + \mu_0\frac{\partial}{\partial t}\vec{j}(\vec{r},t)} \qquad (4)$$

propagation of electric field as function of sources $(\rho,\vec{j})$

Microscopic Maxwell equations

All space is a vacuum. Matter is made of discrete charges q_n (electrons, protons, ..)

LINEAR DIFFERENTIAL EQUATIONS

$$\boxed{\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B} \\ \nabla \times \vec{B} &= \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} \\ \nabla \cdot \vec{E} &= \rho/\varepsilon_0 \\ \nabla \cdot \vec{B} &= 0 \end{aligned}} \tag{4}$$

SI units used throughout course

$[E] = V/m$, $[B] = T = \frac{Vs}{m^2}$

electric field, Tesla, magnetic induction

where all fields are functions of space and time, i.e $\vec{E} = \vec{E}(\vec{r},t)$ etc.

Sources :
$$\boxed{\begin{aligned} \rho(\vec{r},t) &= \sum_n q_n\, \delta(\vec{r} - \vec{r}_n(t)) \\ \vec{j}(\vec{r},t) &= \sum_n q_n \frac{\partial}{\partial t}\vec{r}_n(t)\, \delta(\vec{r} - \vec{r}_n(t)) \end{aligned}} \tag{5}$$

discontinuous functions of position

Each charge q_n experiences a force (c.f. Eq. 3)

$$\boxed{\vec{F}_n = q_n \vec{E}(\vec{r}_n,t) + q_n \frac{\partial}{\partial t}\vec{r}_n(t) \times \vec{B}(\vec{r}_n,t)} \tag{6}$$

Constants :
$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792456\ 10^{8}\ m/s \quad \text{(vacuum speed of light)} \tag{7}$$

$\mu_0 = 4\pi\ 10^{-7}\ \frac{Vs}{Am}$ (permeability)

$\varepsilon_0 = \frac{1}{c^2 \mu_0}\ \frac{A}{V}\frac{s}{m}$ (permittivity)

$e = 1.6021917\ 10^{-19}$ Cb (elementary charge)

Macroscopic Maxwell equations

Fields are averaged over many individual charges. The discrete (atomic) nature of matter is lost. Fields are generated by a 'collective' response of charges (polarization, magnetization). Similar to flow of water: the atomic scale is irrelevant. Macroscopic fields are local spatial averages over microscopic fields

→ charge and current densities are treated as continuous functions of position.

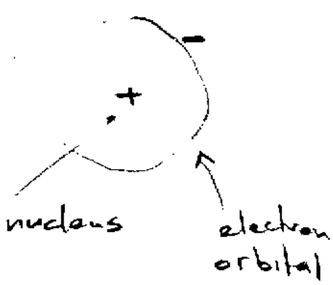
Split the total charge density ρ into two contributions

$$\boxed{\rho = \rho_0 + \rho_{pol}} \tag{8}$$

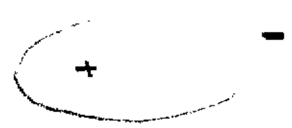
ρ_0 : charge density associated with <u>free</u> charges as introduced by an external mechanism
example: · electron beam in scanning electron microscope
· free electrons in metal

ρ_{pol} : charge density <u>induced</u> in matter through interaction with electromagnetic field.
→ polarization of medium

atomic s-orbital with no external field:

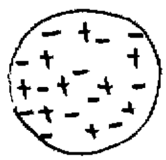


atomic s-orbital with external field:

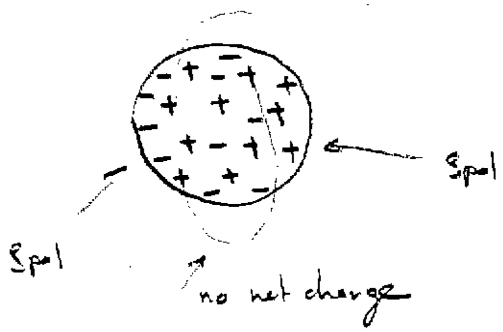


On macroscopic scale:

no external field:



with external field:



Define: $\vec{P}(\vec{r},t)$ = polarization (electric dipole moment per unit volume)

$$\boxed{\nabla\cdot\vec{P} = -\rho_{pol}} \qquad [\vec{P}] = \frac{Cb}{m^2} \tag{9}$$

Insert together with Eq. 8 into Gauss's law:

$$\nabla\cdot(\varepsilon_0\vec{E}+\vec{P}) = \rho_0$$

It is convenient to introduce the <u>displacement</u> $\vec{D}$ as

$$\boxed{\vec{D} = \varepsilon_0\vec{E}+\vec{P}}\,. \qquad [\vec{D}] = \frac{Cb}{m^2} \tag{10}$$

Take time derivative of Eq. 9 $\frac{\partial}{\partial t}\nabla\cdot\vec{P} = -\frac{\partial}{\partial t}\rho_{pol}$, or

$$\nabla\cdot\left[\frac{\partial}{\partial t}\vec{P}\right] = -\frac{\partial}{\partial t}\rho_{pol} \quad . \tag{11}$$

Comparison with Eq. 3 allows us to identify $\frac{\partial}{\partial t}\vec{P}$ as the polarization current density

$$\boxed{\vec{j}_{pol} = \frac{\partial}{\partial t}\vec{P}} \tag{12}$$

Similar to Eq. 8, we split the total current density into two contributions

$$\vec{j} = \vec{j}_0 + \vec{j}_{ind} \,. \tag{13}$$

While the induced charge density could be associated with the polarization charge density generated by electric fields, the induced current density will have both electric and magnetic contributions. An electric field induces charges and currents while a magnetic field gives only rise to currents. Thus, we rewrite Eq. (13) as

$$\boxed{\vec{j} = \vec{j}_0 + \vec{j}_{pol} + \vec{j}_{mag}} \tag{14}$$

$\vec{j}_0$: current density associated with free charges

$\vec{j}_{pol}$: polarization current density

$\vec{j}_{mag}$: magnetization current density

Inserting into the second equation in (4), the 'modified' Ampère law, gives

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \left[\vec{j}_{pol} + \varepsilon_0 \frac{\partial}{\partial t} \vec{E} \right] + \mu_0 \vec{j}_{mag}$$

Using Eq. 12 gives

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \frac{\partial}{\partial t} \underbrace{\left[\vec{P} + \varepsilon_0 \vec{E} \right]}_{= \vec{D}} + \mu_0 \vec{j}_{mag} \tag{15}$$

Define: $\vec{M}(\vec{r},t)$ = magnetization (magnetic dipole moment per unit volume)

$$\boxed{\nabla \times \vec{M} = \vec{j}_{mag}} \qquad [\vec{M}] = \frac{A}{m} \tag{16}$$

Insert into Eq. (15)

$$\nabla \times \vec{B} - \mu_0 \nabla \times \vec{M} = \mu_0 \vec{j} + \mu_0 \frac{\partial}{\partial t} \vec{D} \tag{17}$$

It is convenient to introduce the <u>magnetic field</u> $\vec{H}$ as

$$\boxed{\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}} \, . \qquad [\vec{H}] = \frac{A}{m} \tag{18}$$

The macroscopic Maxwell equations in differential form can now be written as

$$\boxed{\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial}{\partial t} \vec{B} \\ \nabla \times \vec{H} &= \mu_0 \vec{j}_0 + \frac{\partial}{\partial t} \vec{D} \\ \nabla \cdot \vec{D} &= \rho_0 \\ \nabla \cdot \vec{B} &= 0 \end{aligned}} \tag{19}$$

Constitutive relations (always valid)

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \, , \quad \vec{B} = \mu_0 (\vec{H} + \vec{M}) \tag{20}$$

Specific material properties relate $\vec{P}$ to $\vec{E}$ and $\vec{M}$ to $\vec{H}$:

$$\begin{aligned} \vec{P} &\leftrightarrow \vec{E} \\ \vec{M} &\leftrightarrow \vec{H} \end{aligned} \tag{21}$$

$\hookrightarrow$ effects: temporal dispersion
spatial dispersion
nonlinear interactions
⋮