OPT 462: Homework Set 2 Due on Wednesday 31, 2001

1 Spectral representation of fields

1. 1. Determine the spectrum $\hat{\mathbf{E}}(\mathbf{r},\omega)$ of the time harmonic field $\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})\exp(-i\omega t)\}$.

1. 2. Proof the relationship $\hat{\mathbf{E}}(\mathbf{r},-\omega) = \hat{\mathbf{E}}^*(\mathbf{r},\omega)$ for a field with arbitrary time dependence.

2 Transform limited laser pulse

Laser pulses, such as the light from a mode-locked Titanium-Sapphire laser, are commonly described by

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\left\{\mathcal{E}(\mathbf{r},t)\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega_o t)}\right\}\;, \tag{1}$$

where $\mathcal{E}(\mathbf{r},t)$ is a slowly varying envelope vector. It can be written in terms of its amplitude and phase as

$$\mathcal{E}(\mathbf{r},t) \;=\; |\mathcal{E}(\mathbf{r},t)|\;\mathrm{e}^{-i\phi(\mathbf{r},t)}\;. \tag{2}$$

It is convenient to expand the phase into a series as

$$\phi(\mathbf{r},t) \;=\; \phi_o(\mathbf{r}) + \phi_1(\mathbf{r})\,t + \phi_2(\mathbf{r})\,t^2 + \;\ldots\;\;, \tag{3}$$

where ϕ_1 and ϕ_2 are associated with a frequency shift and a linear chirp, respectively.

2. 1. What condition(s) have to hold in order that Eq. 1 satisfies the wave equation in empty space? Is it possible to separate the time dependence?

2. 2. Consider a fixed position $\mathbf{r}=0$ and assume that the envelope function is a Gaussian function

$$\mathcal{E}(\mathbf{r}=0,t) \;=\; \mathbf{E}_o\,\mathrm{e}^{-t^2/(2\tau^2)}\;. \tag{4}$$

Let Δt denote the width of $I(t)=|\mathcal{E}(\mathbf{r}=0,t)|^2$ at 1/e of the maximum value of $I(t)$. Similarily, $\Delta\omega$ is the spectral width of the pulse spectrum $\hat{I}(\omega)$ at 1/e of the maximum value of $\hat{I}(\omega)$. Determine the time-bandwidth product $\Delta t\,\Delta\omega$.
Hint: use the transformation of $\mathcal{E}(t)\exp(-i\omega_o t)$.

2. 3. Allow for a linear chirp $\phi_2(\mathbf{r}=0)=\beta$ and determine $\Delta t\,\Delta\omega$. For what values of β is $\Delta t\,\Delta\omega$ minimized?

2. 4. The unchirped pulse passes through a dispersive medium that is characterized by the electric susceptibility

$$\chi_e(\omega) \;=\; \frac{\omega_p^2}{\omega_n^2 - \omega^2 - i\gamma\omega}\;, \tag{5}$$

with known constants ω_p, ω_n, and γ. To the red side of the resonance $(\omega \ll \omega_n)$ χ_e can be approximated by $\chi_e(\omega)=(\omega_p^2/\omega_n^2)\,(1+\omega^2/\omega_n^2)$. Determine for this case the polarization $\mathbf{P}(\mathbf{r}=0,t)$ and discuss the result.

3 Surface waves

Two homogeneous, linear, isotropic and non-magnetic half-spaces are separated by an interface at $z=0$. The upper half-space $z>0$ is a non-dispersive dielectric with dielectric constant ε_1, whereas the lower half-space is a metal with dielectric constant $\varepsilon_2(\omega)$. We seek solutions of Maxwell's equations of the following form:

$$\mathbf{E}_n(\mathbf{r},t) \;=\; \mathrm{Re}\left\{[E_{n_x}\mathbf{n}_x + E_{n_z}\mathbf{n}_z]\; \mathrm{e}^{i[k_{n_x}x + k_{n_z}z - \omega t]}\right\} \qquad n \in [1,2]\,, \tag{6}$$

where $\mathbf{n}_x$ and $\mathbf{n}_z$ are unit vectors in direction of x and z, respectively.

3. 1. Use Gauss's law to find a relationship between E_{n_x} and E_{n_z} in each half-space.

3. 2. Use the boundary conditions for the electric field to derive an expression for k_x as a function of ω and ε_n (dispersion relation).

3. 3. Derive the magnetic field and proof that it fulfills the boundary conditions.

3. 4. We seek solutions for the electromagnetic field that are bound to the interface. Therefore, the fields have to vanish for $z \to \pm\infty$. Determine a condition for ε_2 in order to make such solutions possible.

3. 5. For a metal with free electrons the dielectric constant reads as

$$\varepsilon_2(\omega) \;=\; 1 - \frac{\omega_p^2}{\omega^2}\,, \tag{7}$$

where ω_p is the plasma frequency. Plot qualitatively the dispersion relation $\omega(k_x)$ and compare it with free propagating light $\omega(k)=k/c$. Discuss the situation $\omega \to \omega_p/\sqrt{1+\varepsilon_1}$ (surface plasmon polariton).