Chapter 14

Electromagnetic Fluctuations

The thermal and zero-point motion of electrically charged particles inside materials gives rise to a fluctuating electromagnetic field. Quantum theory tells us that the fluctuating particles can only assume discrete energy states and, as a consequence, the emitted fluctuating radiation takes on the spectral form of blackbody radiation. However, while the familiar blackbody radiation formula is strictly correct only at thermal equilibrium, it is only an approximation for non-equilibrium situations. This approximation is reasonable at larger distances from the emitting material (farfield) but it can strongly deviate from the true behavior close to material surfaces (near-field).

Because fluctuations of charge and current in materials lead to dissipation via radiation no object at finite temperature can be in thermal equilibrium in free space. Equilibrium with the radiation field can only be achieved by confining the radiation to a finite space. However, in most cases the object can be considered to be close to equilibrium and the non-equilibrium behavior can be described by linear response theory. In this regime, the most important theorem is the *fluctuation-dissipation theorem*. It relates the rate of energy dissipation in a non-equilibrium system to the fluctuations that occur spontaneously at different times in equilibrium systems.

The fluctuation-dissipation theorem is of relevance for the understanding of fluctuating fields near nanoscale objects and optical interactions at nanoscale distances. This chapter is intended to provide a detailed derivation of some important aspects in fluctuational electrodynamics.

14.1 The Fluctuation-Dissipation Theorem

The fluctuation-dissipation theorem is most commonly derived by applying Fermi's Golden Rule and evaluating quantum correlation functions. The theorem has its roots in Nyquist's relation for voltage fluctuations across a resistor. However, it was Callen and Welton who derived the theorem in its general form [1]. The here presented derivation is purely classical. A substitution at the end of the derivation introduces Planck's constant into the theorem. Although the fluctuation-dissipation theorem can be derived in quite general terms using unspecified variables, it is more intuitive to stick to a particular physical situation. Here, we consider a nanoscale system with characteristic dimensions much smaller than the wavelength of light (c.f. Fig. 14.1). This allows us to treat the interaction with the system in the electric dipole approximation. The theory can be easily extended by including higher order multipolar terms. The nanoscale system consists of a finite number of charged particles with N degrees of freedom. At *thermal equilibrium*, the probability for the system's dipole moment $\boldsymbol{\mu}$ to be in state $s = [q_1 \,..\, q_N;\, p_1 \,..\, p_N]$ is given by the distribution function

$$f_{eq}(s) \;=\; f_o \, \mathrm{e}^{-H_o(s)/kT} \;, \tag{14.1}$$

where f_o is a normalization constant ensuring that $\int f_{eq} ds = 1$. H_o is the equilibrium Hamiltonian of the system, k the Boltzmann constant, and T the temperature. q_j and p_j denote the generalized coordinates and conjugate momenta, respectively. s is a point in phase-space and can be viewed as an abbreviation for all the coordinates and momenta of the system. At thermal equilibrium the ensemble average of $\boldsymbol{\mu}$ is

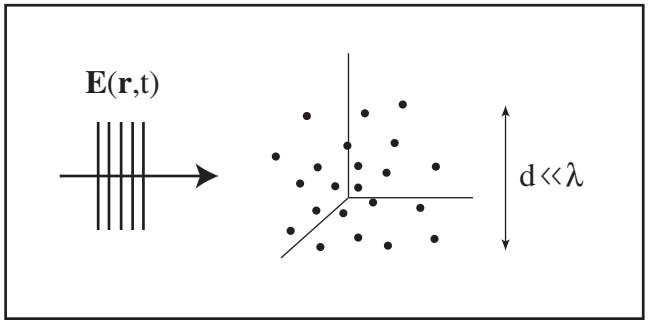

Figure 14.1: Interaction of an optical field with a system of particles initially at thermal equilibrium. The state of the system is defined by the phase-space coordinate $s = [q_1 \,..\, q_N;\, p_1 \,..\, p_N]$, with q_j and p_j being the coordinates and conjugate momenta, respectively. If the characteristic length scale d of the system is small compared with the wavelength λ, the interaction energy between the optical field and the system is given by the electric dipole approximation $\delta H = -\boldsymbol{\mu}(s,t) \cdot \mathbf{E}(t)$, where $\boldsymbol{\mu}$ is the electric dipole moment.

defined as

$$\langle \boldsymbol{\mu}(s,t) \rangle \;=\; \frac{\int f_{eq}(s)\, \boldsymbol{\mu}(s,t)\, ds}{\int f_{eq}(s)\, ds} \;=\; \langle \boldsymbol{\mu} \rangle \;, \tag{14.2}$$

where the integration runs over all coordinates $[q_1 \,..\, q_N;\, p_1 \,..\, p_N]$. Because of equilibrium the ensemble average is independent of time.

14.1.1 The system response function

Let us consider an external field $\mathbf{E}(\mathbf{r},t)$ which perturbs the equilibrium of the system. Assuming that the characteristic dimensions d of the system are much smaller than the wavelength λ we can apply the dipole approximation and the Hamiltonian of the perturbed system becomes

$$H \;=\; H_o + \delta H \;=\; H_o - \boldsymbol{\mu}(s,t)\cdot \mathbf{E}(t) \;=\; H_o - \sum_k \mu_k(s,t) E_k(t) \quad k = x,y,z \;. \tag{14.3}$$

Due to the external perturbation $\mathbf{E}(t)$ the expectation value of $\boldsymbol{\mu}$ will deviate from its equilibrium average $\langle \boldsymbol{\mu} \rangle$. We will designate the expectation value of $\boldsymbol{\mu}$ in the perturbed system by $\bar{\boldsymbol{\mu}}$ in order to distinguish it from $\langle \boldsymbol{\mu} \rangle$. We assume that the deviation

$$\delta \bar{\boldsymbol{\mu}} \;=\; \bar{\boldsymbol{\mu}} - \langle \boldsymbol{\mu} \rangle \tag{14.4}$$

is small and that it linearly depends on the external perturbation, i.e.

$$\delta \bar{\mu}_j(t) \;=\; \frac{1}{2\pi} \sum_k \int_{-\infty}^{t} \tilde{\alpha}_{jk}(t-t')\, E_k(t')\, dt' \qquad j,k = x,y,z \;. \tag{14.5}$$

Here, $\tilde{\alpha}_{jk}$ is the response function of the system. We have assumed that the system is stationary $[\tilde{\alpha}_{jk}(t,t') = \tilde{\alpha}_{jk}(t-t')]$ and causal $[\tilde{\alpha}_{jk}(t-t') = 0 \text{ for } t' > t]$. Eq. (14.5)

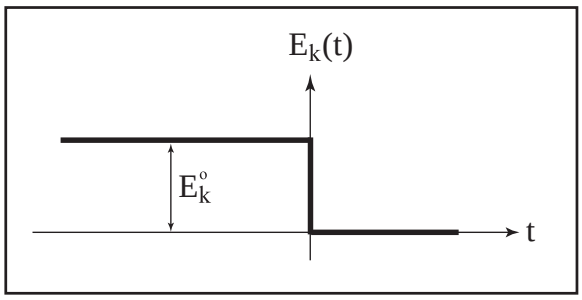

Figure 14.2: Time-dependence of the considered perturbation. The perturbation ensures complete relaxation of the system at times $t = 0$ (immediately before the step) and $t \to \infty$.

states that the response at time t not only depends on the perturbation at time t but also on the perturbations prior to t. The "memory" of the system is contained in $\tilde{\alpha}_{jk}$. Our goal is to determine $\tilde{\alpha}_{jk}$ as a function of the statistical equilibrium properties of the system. It is convenient to consider the perturbation shown in Fig. 14.2 which promotes the system from one completely relaxed (equilibrated) state to another [2]. The relaxation time can be intuitively associated with the memory of the response function. Evaluating Eq. (14.5) for the perturbation shown in Fig. 14.2 gives

$$\delta\bar{\mu}_j(t) \;=\; \frac{E_k^o}{2\pi}\int\limits_{-\infty}^{0}\tilde{\alpha}_{jk}(t-t')\,dt' \;=\; \frac{E_k^o}{2\pi}\int\limits_{t}^{\infty}\tilde{\alpha}_{jk}(\tau)\,d\tau\;, \tag{14.6}$$

which can be solved for $\tilde{\alpha}_{jk}$ as

$$\tilde{\alpha}_{jk}(t) \;=\; -\frac{2\pi}{E_k^o}\,\Theta(t)\,\frac{d}{dt}\delta\bar{\mu}_j(t)\;. \tag{14.7}$$

Here, we assumed that $\tilde{\alpha}_{jk}$ and its time-derivative tend to zero for times $t\to\infty$ and we introduced the Heaviside step-function $\Theta(t)$ to ensure causality [$\tilde{\alpha}_{jk}(t-t')=0$ for $t'>t$]*. According to Eq. (14.7), we find $\tilde{\alpha}_{jk}$ if we calculate the time-derivative of $\delta\bar{\mu}_j$ at time t.

The expectation value of $\boldsymbol{\mu}$ at time t is determined by the distribution function $f(s)$ at the initial time $t=0$ according to (c.f. Fig. 14.3)

$$\bar{\boldsymbol{\mu}}(t) \;=\; \frac{\int f(s)\,\boldsymbol{\mu}(s,t)\,ds}{\int f(s)\,ds}\;. \tag{14.8}$$

*$\Theta(t)=0$ for $t<0$, $\Theta(t)=1/2$ for $t=0$, and $\Theta(t)=1$ for $t>0$.

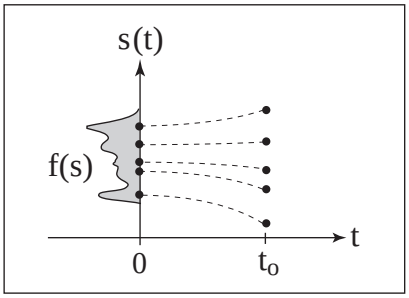

Figure 14.3: Newton's equations of motion map each phase-space point s at time $t=0$ into a phase-space point at time t_o. The dipole moment at time t_o can be expressed as $\boldsymbol{\mu}[s(t_o)]=\boldsymbol{\mu}[s(0),t_o]=\boldsymbol{\mu}[s,t_o]$ and its ensemble average at time t_o is determined by the initial distribution function $f(s)$.

Because of thermal equilibrium at time $t=0$, the distribution function reads as

$$f(s) \,\propto\, \mathrm{e}^{-[H_o+\delta H]/kT} \,=\, f_{eq}(s)\,\mathrm{e}^{-\delta H(s)/kT} \,=\, f_{eq}(s)\left[1-\frac{1}{kT}\,\delta H(s)+..\right]\,, \quad (14.9)$$

where $f_{eq}(s)$ is given by Eq. (14.1). The last term in brackets is the series expansion of $\exp(-\delta H/kT)$. Inserting into Eq. (14.8) and retaining only terms up to linear order in δH we obtain[†]

$$\bar{\boldsymbol{\mu}}(t) \,=\, \Big\langle \boldsymbol{\mu} \Big\rangle \,-\, \frac{1}{kT}\left[\Big\langle \delta H(s)\,\boldsymbol{\mu}(s,t)\Big\rangle \,-\, \Big\langle \boldsymbol{\mu}(s,t)\Big\rangle\Big\langle \delta H(s)\Big\rangle\right]\,, \quad (14.10)$$

where $\langle..\rangle$ denotes the expectation value in the absence of the perturbation, i.e. the expectation value calculated by using the distribution function f_{eq} in Eq. (14.1). Since $\delta H(s)$ is the perturbation at time $t=0$ we have $\delta H(s) = -\mu_k(s,0)\,E_k^o$ and Eq. (14.10) can be rewritten as

$$\begin{aligned}\delta\bar{\mu}_j(t) \,&=\, \bar{\mu}_j(t) - \langle\mu_j\rangle \,=\, -\frac{E_k^o}{kT}\left[\Big\langle\mu_j\Big\rangle\Big\langle\mu_k\Big\rangle - \Big\langle\mu_k(0)\,\mu_j(t)\Big\rangle\right] \quad (14.11)\\ &=\, \frac{E_k^o}{kT}\Big\langle\big[\mu_k(0)-\langle\mu_k\rangle\big]\big[\mu_j(t)-\langle\mu_j\rangle\big]\Big\rangle \,=\, \frac{E_k^o}{kT}\Big\langle\delta\mu_k(0)\,\delta\mu_j(t)\Big\rangle\,,\end{aligned}$$

where we used Eq. (14.2) and defined $\delta\bar{\mu}_j(t) = [\mu_j(t) - \langle\mu_j\rangle]$. Introducing this result into Eq. (14.7) we finally find

$$\boxed{\tilde{\alpha}_{jk}(t) \,=\, -\frac{2\pi}{kT}\,\Theta(t)\,\frac{d}{dt}\Big\langle\delta\mu_k(0)\,\delta\mu_j(t)\Big\rangle} \quad (classical)\,. \quad (14.12)$$

This important result is often referred to as the time-domain fluctuation dissipation theorem. It states that the system's response to a weak external field can be expressed in terms of the system's fluctuations in the absence of the external field! Notice that the correlation function $\langle\delta\mu_k(0)\,\delta\mu_j(t)\rangle$ is a property of the stationary equilibrium system and that the correlation function can be offset by an arbitrary time τ as

$$\Big\langle\delta\mu_k(0)\,\delta\mu_j(t)\Big\rangle \,=\, \Big\langle\delta\mu_k(\tau)\,\delta\mu_j(t+\tau)\Big\rangle\,. \quad (14.13)$$

For many problems it is convenient to express Eq. (14.12) in the frequency domain by using the Fourier transforms[‡]

$$\alpha_{jk}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\alpha}_{jk}(t)\,\mathrm{e}^{i\omega t}\,dt\,, \quad \delta\hat{\mu}_j(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta\mu_j(t)\,\mathrm{e}^{i\omega t}\,dt\,. \quad (14.14)$$

[†] $[1-\langle\delta H\rangle/kT]^{-1} \approx\, [1+\langle\delta H\rangle/kT\,-\,..\,]$

[‡] Because the function $\delta\mu_j(t)$ is a stochastic process it is not square integrable and therefore its Fourier transform is not defined. However, these difficulties can be overcome by the theory of generalized functions and it can be shown that the Fourier transform can be used in symbolic form [7].

The correlation function in frequency domain $\langle \delta\hat{\mu}_j(\omega)\, \delta\hat{\mu}_k^*(\omega')\rangle$ can be calculated by substituting the Fourier transforms for $\delta\hat{\mu}_j(\omega)$ and $\delta\hat{\mu}_k^*(\omega')$ as

$$\begin{aligned}\left\langle \delta\hat{\mu}_j(\omega)\, \delta\hat{\mu}_k^*(\omega')\right\rangle &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\left\langle \delta\mu_j(\tau')\, \delta\mu_k(\tau)\right\rangle \mathrm{e}^{i[\omega\tau'-\omega'\tau]}\, d\tau'\, d\tau \\ &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\left\langle \delta\mu_k(\tau)\, \delta\mu_j(t+\tau)\right\rangle \mathrm{e}^{i[\omega-\omega']\tau}\ \mathrm{e}^{i\omega t}\, d\tau\, dt\,, \quad (14.15)\end{aligned}$$

where we used the substitution $\tau' = \tau + t$. Because of stationarity the correlation function in the integrand does not depend on τ and the integration over τ reduces to a delta-function [§]. The final relation is known as the *Wiener-Khintchine theorem*

$$\left\langle \delta\hat{\mu}_j(\omega)\, \delta\hat{\mu}_k^*(\omega')\right\rangle = \delta(\omega-\omega')\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle \delta\mu_k(\tau)\, \delta\mu_j(t+\tau)\right\rangle \mathrm{e}^{i\omega t}\, dt\,, \qquad (14.16)$$

which demonstrates that spectral components that belong to different frequencies are uncorrelated. The integral on the right hand side is known as the *spectral density*. To obtain a spectral representation of the fluctuation-dissipation theorem, we need to Fourier transform Eq. (14.12). The right hand-side leads to a convolution between the spectrum of the step function, $\hat{\Theta}(\omega)$,[¶] and the spectrum of $d/dt\langle\delta\mu_k(0)\delta\mu_j(t)\rangle$. To get rid of the imaginary part of $\hat{\Theta}$ we solve for $[\alpha_{jk}(\omega)-\alpha_{kj}^*(\omega)]$ instead of $\alpha_{jk}(\omega)$. Making use of stationarity, the Wiener-Khintchine theorem, and the fact that $\langle \delta\mu_k(\tau)\, \delta\mu_j(t+\tau)\rangle$ is real, we obtain

$$\left[\alpha_{jk}(\omega) - \alpha_{kj}^*(\omega)\right]\delta(\omega-\omega') = \frac{2\pi i\omega}{kT}\left\langle \delta\hat{\mu}_j(\omega)\, \delta\hat{\mu}_k^*(\omega')\right\rangle \quad (classical)\,. \qquad (14.17)$$

This is the analogue of Eq. (14.12) in frequency domain. The factor kT can be identified as the average energy per degree of freedom of a particle in the system (equipartition principle). This average energy is based on the assumption that the energy distribution of electromagnetic modes is continuous. However, according to quantum mechanics these modes can only assume discrete energy values separated by $\delta E = \hbar\omega$ and, as a consequence, the average energy kT has to be substituted as

$$kT \;\rightarrow\; \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} + \hbar\omega\,, \qquad (14.18)$$

which corresponds to the mean energy of the quantum oscillator (first term) plus the zero point energy $\hbar\omega$ (second term). We choose $\hbar\omega$ instead of $\hbar\omega/2$ in order to be consistent with quantum theory which requires that $\left\langle \delta\hat{\mu}_j(\omega)\, \delta\hat{\mu}_k^*(\omega')\right\rangle$ is an anti-normally ordered quantity for $\omega > 0$ (c.f. Section 14.1.4).

[§] $\int_{-\infty}^{\infty}\exp(ixy)dy = 2\pi\delta(x)$.

[¶] $\hat{\Theta}(\omega) = \frac{1}{2}\delta(\omega) - \frac{1}{2\pi}\frac{1}{i\omega}$.

In the limit $\hbar \to 0$ or $\hbar\omega \ll kT$ the substitution 14.18 recovers the classical value of kT. Rewriting the right hand side of Eq. (14.18) as $\hbar\omega/(1-\exp[-\hbar\omega/kT])$ and substituting into Eq. (14.17) renders the quantum version of the fluctuation-dissipation theorem [8, 12]

$$\boxed{\left\langle \delta\hat{\mu}_j(\omega)\,\delta\hat{\mu}_k^*(\omega')\right\rangle \;=\; \frac{1}{2\pi i\omega}\left[\frac{\hbar\omega}{1\;-\;\mathrm{e}^{-\hbar\omega/kT}}\right]\Big[\alpha_{jk}(\omega)-\alpha_{kj}^*(\omega)\Big]\delta(\omega-\omega')} \tag{14.19}$$

While *dissipation* is associated with the right hand side, the left hand side represents *fluctuations* of the equilibrium system. It is important to notice that quantum mechanics leads to dissipation even for temperatures at absolute zero. The remaining fluctuations affect only *positive* frequencies! This can easily be seen by the following limit

$$\lim_{T\to 0}\left[\frac{1}{1\;-\;\mathrm{e}^{-\hbar\omega/kT}}\right] \;=\; \Theta(\omega) \;=\; \begin{cases} 1 & \omega>0 \\ 1/2 & \omega=0 \\ 0 & \omega<0 \end{cases} \tag{14.20}$$

The fluctuation-dissipation theorem can be generalized to include the spatial dependence of the sources. It turns out that as long as the system's response function is local, i.e. $\tilde{\varepsilon}_{jk}(\mathbf{r},t)=\tilde{\varepsilon}_{jk}(t)$ or $\varepsilon_{jk}(\mathbf{k},\omega)=\varepsilon_{jk}(\omega)$, fluctuations at two distinct spatial coordinates are *uncorrelated* [3]. For a fluctuating current density $\delta\mathbf{j}(\mathbf{r},t)$ in an *isotropic* and *homogeneous* medium with dielectric constant $\varepsilon(\omega)$, Eq. (14.19) can be generalized as [4]

$$\left\langle \hat{\delta j}_j(\mathbf{r},\omega)\,\hat{\delta j}_k^*(\mathbf{r}',\omega')\right\rangle \;=\; \frac{\omega\varepsilon_o}{\pi}\,\varepsilon''(\omega)\left[\frac{\hbar\omega}{1\;-\;\mathrm{e}^{-\hbar\omega/kT}}\right]\,\delta(\omega-\omega')\,\delta(\mathbf{r}-\mathbf{r}')\,\delta_{jk}\;. \tag{14.21}$$

ε'' is the imaginary part of ε, $\hat{\delta j}$ denotes the Fourier transform of δj, and the Kronecker delta δ_{jk} is a consequence of *isotropy*.

14.1.2 Johnson Noise

We finally write the fluctuation-dissipation theorem in the form originally developed by Callen and Welton [1]. We note that the fluctuating dipole moment $\delta\boldsymbol{\mu}$ gives rise to a local stochastic electric field $\delta\mathbf{E}$ according to

$$\delta\hat{\mu}_j(\omega) \;=\; \sum_k \alpha_{jk}(\omega)\,\delta\hat{E}_k(\omega) \qquad j,k=x,y,z\;, \tag{14.22}$$

which directly follows from the time-domain relationship of Eq. (14.5) by using the definitions of Fourier transforms in Eq. (14.14). Substituting the linear relationship into Eq. (14.19) leads to

$$\left\langle \delta\hat{E}_j(\omega)\,\delta\hat{E}_k^*(\omega')\right\rangle \;=\; \frac{1}{2\pi i\omega}\left[\frac{\hbar\omega}{1\;-\;\mathrm{e}^{-\hbar\omega/kT}}\right]\Big[\alpha_{kj}^{*\,-1}(\omega)-\alpha_{jk}^{-1}(\omega)\Big]\,\delta(\omega-\omega')\;. \tag{14.23}$$

This equation renders the local electric field correlation *induced* by the fluctuating dipole. Integrating on both sides over ω' and application of the Wiener-Khintchine theorem leads to

$$\frac{1}{2\pi i\omega}\left[\frac{\hbar\omega}{1\,-\,\mathrm{e}^{-\hbar\omega/kT}}\right]\,\left[\alpha^{*\,-1}_{kj}(\omega)-\alpha^{-1}_{jk}(\omega)\right]\;=\;\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big\langle\delta E_k(\tau)\,\delta E_j(t+\tau)\Big\rangle\,\mathrm{e}^{i\omega t}\,dt\;. \tag{14.24}$$

Further integration over ω gives rise to a delta-function on the right hand side which allows the time-integral to be evaluated. The final result reads as

$$\Big\langle\delta E_k(\tau)\,\delta E_j(\tau)\Big\rangle\;=\;\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{1}{i\omega}\left[\frac{\hbar\omega}{1\,-\,\mathrm{e}^{-\hbar\omega/kT}}\right]\,\left[\alpha^{*\,-1}_{kj}(\omega)-\alpha^{-1}_{jk}(\omega)\right]d\omega\;. \tag{14.25}$$

We now apply this formula to charge fluctuations in a resistor. The fluctuating current density can be expressed in terms of the fluctuating dipole moment as $\delta j = d/dt[\delta\mu]\delta(\mathbf{r}-\mathbf{r}')$. Assuming an isotropic resistor ($j=k$) the relationship between current and field in spectral domain becomes $\delta\hat{j}(\omega)=-i\omega\alpha(\omega)\delta(\mathbf{r}-\mathbf{r}')\delta\hat{E}$ which allows us to identify the term $[-i\omega\alpha(\omega)\delta(\mathbf{r}-\mathbf{r}')]^{-1}$ with the resistivity $\rho(\omega)$. Assuming that $\rho(\omega)$ is real, we can express Eq. (14.25) as

$$\Big\langle\delta E^2\Big\rangle\;=\;\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{\hbar\omega}{1\,-\,\mathrm{e}^{-\hbar\omega/kT}}\right]\,\rho(\omega)\,\delta(\mathbf{r}-\mathbf{r}')\,d\omega\;, \tag{14.26}$$

which can be rewritten in terms of the voltage V and resistance R as

$$\begin{aligned}\Big\langle\delta V^2\Big\rangle\;&=\;\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{\hbar\omega}{1\,-\,\mathrm{e}^{-\hbar\omega/kT}}\right]R(\omega)\,d\omega\\ &=\;\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{\hbar\omega}{1\,-\,\mathrm{e}^{-\hbar\omega/kT}}\right]R(\omega)\,d\omega\;-\;\frac{1}{\pi}\int_{0}^{\infty}\left[\frac{\hbar\omega}{1\,-\,\mathrm{e}^{-\hbar\omega/kT}}\;-\;\hbar\omega\right]R(-\omega)\,d\omega\\ &=\;\frac{2}{\pi}\int_{0}^{\infty}\left[\frac{\hbar\omega}{\mathrm{e}^{\hbar\omega/kT}\,-\,1}\;+\;\frac{1}{2}\hbar\omega\right]R(\omega)\,d\omega\;.\end{aligned} \tag{14.27}$$

We reduced the integration range to $[0\,..\,\infty]$ and made use of $R(\omega)=-R(-\omega)$. The left hand side can be identified with the mean square voltage fluctuations. For temperatures $kT\gg\hbar\omega$, which is fulfilled for any practical frequencies at room temperature, we can replace the expression in brackets by its classical limit kT. Furthermore, for a system with finite bandwidth $B=(\omega_{max}-\omega_{min})/2\pi$ and a frequency independent resistance we obtain

$$\Big\langle\delta V^2\Big\rangle\;=\;4kTBR\;. \tag{14.28}$$

This is the familar formula for *white noise*, also called *Johnson noise*, generated in electrical circuits by resistors. In a bandwidth of $10kHz$ and at room temperature, a resistor of $10M\Omega$ generates a voltage of $\approx 40\mu V_{rms}$.

14.1.3 Dissipation due to fluctuating external fields

We have derived the dissipation of a system as a function of its charge fluctuations. Here we intend to express dissipation in terms of the fields that the fluctuating charges generate. Each current density $\delta\hat{\mathbf{j}}$ in Eq. (14.21) generates an electric field according to

$$\delta\hat{\mathbf{E}}(\mathbf{r},\omega) \;=\; i\omega\mu_o \int\limits_{V_o} \overleftrightarrow{\mathbf{G}}\,(\mathbf{r},\mathbf{r}_o;\omega)\,\delta\hat{\mathbf{j}}(\mathbf{r}_o,\omega)\;d^3\mathbf{r}_o\;, \tag{14.29}$$

where all currents are confined in the source region V_o. Multiplying the above expression with the corresponding expression for the field $\delta\hat{\mathbf{E}}(\mathbf{r}',\omega')$, taking the ensemble average, and applying Eq. (14.21) gives

$$\begin{aligned}\left\langle \delta\hat{E}_j(\mathbf{r},\omega)\,\delta\hat{E}_k^*(\mathbf{r}',\omega')\right\rangle \;&=\; \frac{\omega^3}{\pi c^4\varepsilon_o}\left[\frac{\hbar\omega}{1\;-\;\mathrm{e}^{-\hbar\omega/kT}}\right]\,\delta(\omega-\omega')\;\times \\ &\sum_n \int_{V_o} G_{jn}(\mathbf{r},\mathbf{r}_o;\omega)\,\varepsilon''(\omega)\,G_{kn}(\mathbf{r}',\mathbf{r}_o;\omega)\;d^3\mathbf{r}_o\;.\end{aligned} \tag{14.30}$$

We now note that the dielectric properties of the source region are not only defined by ε'' but also by $\overleftrightarrow{\mathbf{G}}$ because its definition depends on the factor $k^2=(\omega/c)^2\,\varepsilon(\omega)$ [c.f. Eq. (**??**)]. Therefore, it is possible to rewrite the above equation for the electric field correlations by using the identity [5, 6]

$$\sum_n \int\limits_{V_o} G_{jn}(\mathbf{r},\mathbf{r}_o;\omega)\,\varepsilon''(\omega)\,G_{kn}(\mathbf{r}',\mathbf{r}_o;\omega)\;d^3\mathbf{r}_o \;=\; \mathrm{Im}\left\{G_{jk}(\mathbf{r},\mathbf{r}';\omega)\right\}\;, \tag{14.31}$$

which can be derived by using $G_{ij}(\mathbf{r}',\mathbf{r};\omega)=G_{ji}(\mathbf{r},\mathbf{r}';\omega)$, requiring that the Green's function is zero at infinity, and by making use of the definition of $\overleftrightarrow{\mathbf{G}}$ [Eq. (**??**)]. In order for $\overleftrightarrow{\mathbf{G}}$ to be zero at infinity, $\overleftrightarrow{\mathbf{G}}$ has to consist of an outgoing and an incoming part ensuring that there is no net energy transport, i.e. the time-averaged Poynting vector has to be zero for any point in space. This condition ensures that all charges are in equilibrium with the radiation field [9].

The fluctuation dissipation theorem for the electric field can now be expressed in terms of the Green's function alone as

$$\boxed{\left\langle \delta\hat{E}_j(\mathbf{r},\omega)\,\delta\hat{E}_k^*(\mathbf{r}',\omega')\right\rangle = \frac{\omega}{\pi c^2\varepsilon_o}\left[\frac{\hbar\omega}{1-\mathrm{e}^{-\hbar\omega/kT}}\right]\mathrm{Im}\{G_{jk}(\mathbf{r},\mathbf{r}';\omega)\}\;\delta(\omega-\omega')} \tag{14.32}$$

This result establishes the correspondence between field fluctuations (left side) and dissipation (right side) which is expressed in terms of the imaginary part of the Green's function. As before, the result is strictly valid only at equilibrium, i.e. when the field and the sources are at the same temperature.

14.1.4 Normal and antinormal ordering

Let us split the electric field $\mathbf{E}(t)$ at an arbitrary space point $\mathbf{r}$ into two parts as

$$\mathbf{E}(t) \;=\; \mathbf{E}^{+}(t) \,+\, \mathbf{E}^{-}(t) \;=\; \int\limits_{0}^{\infty} \hat{\mathbf{E}}(\omega)\, \mathrm{e}^{-i\omega t}\, d\omega \;+\; \int\limits_{-\infty}^{0} \hat{\mathbf{E}}(\omega)\, \mathrm{e}^{-i\omega t}\, d\omega \;, \qquad (14.33)$$

where $\hat{\mathbf{E}}(\omega)$ is the Fourier spectrum of $\mathbf{E}(t)$. The functions $\mathbf{E}^{+}$ and $\mathbf{E}^{-}$ are no longer real functions but so-called complex analytical signals [7]. $\mathbf{E}^{+}$ is defined by the *positive* frequencies of $\hat{\mathbf{E}}$ whereas $\mathbf{E}^{-}$ is defined by the *negative* frequencies of $\hat{\mathbf{E}}$. Because $\mathbf{E}(t)$ is real we have $\hat{\mathbf{E}}^{*}(\omega) = \hat{\mathbf{E}}(-\omega)$ which implies that $\mathbf{E}^{-} = [\mathbf{E}^{+}]^{*}$. Let us also define the (inverse) Fourier transforms of $\mathbf{E}^{+}$ and $\mathbf{E}^{-}$:

$$\mathbf{E}^{+}(t) = \int\limits_{-\infty}^{\infty} \hat{\mathbf{E}}^{+}(\omega)\, \mathrm{e}^{-i\omega t}\, d\omega \;, \quad \mathbf{E}^{-}(t) = \int\limits_{-\infty}^{\infty} \hat{\mathbf{E}}^{-}(\omega)\, \mathrm{e}^{-i\omega t}\, d\omega \;. \qquad (14.34)$$

Obviously, the spectra are related to the original spectrum $\hat{\mathbf{E}}$ as

$$\hat{\mathbf{E}}^{+}(\omega) \;=\; \begin{cases} \hat{\mathbf{E}}(\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases} \;, \qquad \hat{\mathbf{E}}^{-}(\omega) \;=\; \begin{cases} 0 & \omega > 0 \\ \hat{\mathbf{E}}(\omega) & \omega < 0 \end{cases} \;. \qquad (14.35)$$

In quantum mechanics, $\hat{\mathbf{E}}^{-}$ is associated with the creation operator $\hat{a}^{\dagger}$ and $\hat{\mathbf{E}}^{+}$ with the annihilation operator $\hat{a}$ (c.f. Section **??**). The sequence $\hat{\mathbf{E}}^{-}\hat{\mathbf{E}}^{+}$ describes the probability for photon absorption and the sequence $\hat{\mathbf{E}}^{+}\hat{\mathbf{E}}^{-}$ the probability for photon emission [7]. The important thing is that in quantum mechanics the two processes are not the same, i.e. $\hat{\mathbf{E}}^{+}$ and $\hat{\mathbf{E}}^{-}$ do not commute. Therefore, we need to calculate separately the correlations of $\hat{\mathbf{E}}^{-}\hat{\mathbf{E}}^{+}$ (normal ordering) and $\hat{\mathbf{E}}^{+}\hat{\mathbf{E}}^{-}$ (antinormal ordering).

We now draw our attention to the fluctuating field $\delta\hat{\mathbf{E}}(\mathbf{r}, t)$ with zero average value and we decompose its Fourier spectrum into positive and negative frequency parts. Using the results from Ref. [8] and procedures similar to Eq. (14.32) we find

$$\left\langle \delta\hat{E}_j^{-}(\mathbf{r},\omega)\; \delta\hat{E}_k^{+^{*}}(\mathbf{r}',\omega') \right\rangle \;=\; \frac{\omega\,\Theta(-\omega)}{\pi c^2 \varepsilon_o} \left[\frac{\hbar\omega}{1 \,-\, \mathrm{e}^{-\hbar\omega/kT}} \right] \mathrm{Im}\left\{ G_{jk}(\mathbf{r},\mathbf{r}';\omega) \right\}\, \delta(\omega - \omega') \qquad (14.36)$$

$$\left\langle \delta\hat{E}_j^{+}(\mathbf{r},\omega)\; \delta\hat{E}_k^{-^{*}}(\mathbf{r}',\omega') \right\rangle \;=\; \frac{\omega\,\Theta(\omega)}{\pi c^2 \varepsilon_o} \left[\frac{\hbar\omega}{1 \,-\, \mathrm{e}^{-\hbar\omega/kT}} \right] \mathrm{Im}\left\{ G_{jk}(\mathbf{r},\mathbf{r}';\omega) \right\}\, \delta(\omega - \omega') \qquad (14.37)$$

where $\Theta(\omega)$ is the unit step function. Hence, the correlation of the normally ordered operators is zero for positive frequencies. Similarly, the correlation of the antinormally ordered operators is zero for negative frequencies.

It can be shown that $\left\langle \delta\hat{E}_j^{-}\; \delta\hat{E}_k^{-^{*}} \right\rangle = \left\langle \delta\hat{E}_j^{+}\; \delta\hat{E}_k^{+^{*}} \right\rangle = 0$, and hence the correlations for the total field $\hat{E} = \hat{E}^{-} + \hat{E}^{+}$ are simply the sum of the correlations for the normally

and antinormally ordered fields given above. This recovers our result Eq. (14.32) and allows us to interpret the correlation $\langle \delta\hat{E}_j \, \delta\hat{E}_k^* \rangle$ as a sequence of absorption and emission events.

For completness, we also state the fluctuation dissipation theorem for symmetrized correlation functions. The quantity of interest is

$$\frac{1}{2}\Big\langle \Big[\delta\hat{E}_j(\mathbf{r},\omega)\, \delta\hat{E}_k^*(\mathbf{r}',\omega') \;+\; \delta\hat{E}_k(\mathbf{r},\omega)\, \delta\hat{E}_j^*(\mathbf{r}',\omega') \Big] \Big\rangle \,. \tag{14.38}$$

Using Eq. (14.36) and (14.37) it is straightforward to show that the above expression equals

$$\frac{\omega}{\pi c^2 \varepsilon_o}\, \hbar\omega \left[\frac{1}{2} \;+\; \frac{1}{\mathrm{e}^{\hbar\omega/kT} - 1} \right] \mathrm{Im}\left\{ G_{jk}(\mathbf{r},\mathbf{r}';\omega) \right\} \, \delta(\omega - \omega') \,. \tag{14.39}$$

Thus, the only difference compared with Eq. (14.32) is the replacement of a factor 1 by 1/2. Consequently, for $T = 0$ the symmetrized correlations are no longer zero at negative frequencies.

14.2 Emission by Fluctuating Sources

The energy density of an arbitrary fluctuating electromagnetic field in vacuum is given by [c.f. Eq. (**??**)]

$$W(\mathbf{r},t) \;=\; \frac{\varepsilon_o}{2}\, \delta\mathbf{E}(\mathbf{r},t)\cdot \delta\mathbf{E}(\mathbf{r},t) \;+\; \frac{\mu_o}{2}\, \delta\mathbf{H}(\mathbf{r},t)\cdot \delta\mathbf{H}(\mathbf{r},t) \,, \tag{14.40}$$

For simplicity, we will skip the position vector $\mathbf{r}$ in the arguments. Assuming stationary fluctuations, the average of W becomes

$$\bar{W} \;=\; \int_{-\infty}^{\infty} \bar{W}_\omega(\omega)\, d\omega \;=\; \frac{\varepsilon_o}{2}\Big\langle \delta\mathbf{E}(t)\cdot \delta\mathbf{E}(t) \Big\rangle \;+\; \frac{\mu_o}{2}\Big\langle \delta\mathbf{H}(t)\cdot \delta\mathbf{H}(t) \Big\rangle \,. \tag{14.41}$$

The mean-square value of $\delta\mathbf{E}$ can be expressed as

$$\Big\langle \delta\mathbf{E}(t)\cdot \delta\mathbf{E}(t) \Big\rangle \;=\; \frac{1}{2\pi} \iint\limits_{-\infty}^{\infty} \Big\langle \delta\mathbf{E}(t)\cdot \delta\mathbf{E}(t+\tau) \Big\rangle \, \mathrm{e}^{i\omega\tau}\, d\omega\, d\tau \,, \tag{14.42}$$

with a similar expression for $\delta\mathbf{H}$. We can now identify the spectral energy density $\bar{W}_\omega$ in Eq. (14.41) as[‖]

$$\bar{W}_\omega(\omega) \;=\; \int\limits_{-\infty}^{\infty} \Big[\frac{\varepsilon_o}{4\pi}\Big\langle \delta\mathbf{E}(t)\cdot \delta\mathbf{E}(t+\tau) \Big\rangle \;+\; \frac{\mu_o}{4\pi}\Big\langle \delta\mathbf{H}(t)\cdot \delta\mathbf{H}(t+\tau) \Big\rangle \Big] \, \mathrm{e}^{i\omega\tau}\, d\tau \,. \tag{14.43}$$

[‖] Keep in mind that $\bar{W}_\omega$ is defined for positive and negative frequencies.

After multiplication on both sides with $\delta(\omega–\omega')$, making use of the Wiener-Khintchine theorem [c.f. Eq. (14.16)] and reintroducing the spatial dependence we obtain

$$\bar{W}_\omega(\mathbf{r},\omega)\delta(\omega–\omega') = \frac{\varepsilon_o}{2}\Big\langle \delta\hat{\mathbf{E}}^*(\mathbf{r},\omega)\cdot\delta\hat{\mathbf{E}}(\mathbf{r},\omega')\Big\rangle + \frac{\mu_o}{2}\Big\langle \delta\hat{\mathbf{H}}^*(\mathbf{r},\omega)\cdot\delta\hat{\mathbf{H}}(\mathbf{r},\omega')\Big\rangle \ , \quad (14.44)$$

where $\delta\hat{\mathbf{E}}$ and $\delta\hat{\mathbf{H}}$ are the Fourier transforms of $\delta\mathbf{E}$ and $\delta\mathbf{H}$, respectively. In the farfield, $|\delta\hat{\mathbf{H}}| = |\delta\hat{\mathbf{E}}|\sqrt{\varepsilon_o/\mu_o}$ and the electric and magnetic energy densities turn out to be equal.

We like to determine the spectral energy density $\bar{W}_\omega$ due to a distribution of fluctuating currents $\delta\mathbf{j}$ in an arbitrary polarizable reference system. We assume that the latter can be accounted for by a dyadic Green's function $\overleftrightarrow{\mathbf{G}}\ (\mathbf{r},\mathbf{r}',\omega)$. Using the volume-integral equations discussed in Section **??** we obtain

$$\delta\hat{\mathbf{E}}(\mathbf{r},\omega) \ = \ i\omega\mu_o\int_V \overleftrightarrow{\mathbf{G}}\ (\mathbf{r},\mathbf{r}',\omega)\ \delta\hat{\mathbf{j}}(\mathbf{r}',\omega)\ dV' \quad (14.45)$$

$$\delta\hat{\mathbf{H}}(\mathbf{r},\omega) \ = \ \int_V \Big[\nabla\times\ \overleftrightarrow{\mathbf{G}}\ (\mathbf{r},\mathbf{r}',\omega)\Big]\ \delta\hat{\mathbf{j}}(\mathbf{r}',\omega)\ dV' \ , \quad (14.46)$$

After introducing these equations into the expression for $\bar{W}_\omega$, the averages over the fields reduce to averages over the currents.** The latter can then be eliminated by using the fluctuation-dissipation theorem given in Eq. (14.21). Integration over ω' leads to

$$\bar{W}_\omega(\mathbf{r},\omega) \ = \ \frac{\omega}{\pi\, c^2}\left[\frac{\hbar\omega}{1\ -\ \mathrm{e}^{-\hbar\omega/kT}}\right]\sum_{j,k}\ \times \quad (14.47)$$

$$\int_V \varepsilon''(\mathbf{r}',\omega)\left[\frac{\omega^2}{c^2}\Big|[\overleftrightarrow{\mathbf{G}}\ (\mathbf{r},\mathbf{r}',\omega)]_{jk}\Big|^2 + \Big|[\nabla\times\overleftrightarrow{\mathbf{G}}\ (\mathbf{r},\mathbf{r}',\omega)]_{jk}\Big|^2\right] dV' \ ,$$

with $[\overleftrightarrow{\mathbf{G}}]_{jk}$ and $[\nabla\times\overleftrightarrow{\mathbf{G}}]_{jk}$ denoting the $jk-th$ elements of the tensors $\overleftrightarrow{\mathbf{G}}$ and $(\nabla\times\overleftrightarrow{\mathbf{G}})$, respectively. The first term in the brackets originates from the electric contribution to $\bar{W}_\omega$ whereas the second term is due to the magnetic field. In general, the result for $\bar{W}_\omega$ can be written in the form

$$\bar{W}_\omega(\mathbf{r},\omega) \ = \ \bar{w}(\omega,T)\, N(\mathbf{r},\omega) \ , \quad (14.48)$$

where $\bar{w}(\omega,T)$ is the average energy per mode. $N(\mathbf{r},\omega)$ depends only on the dielectric properties $\varepsilon(\omega)$ and the Green's function of the reference system. It has a similar meaning than the *local density of states* defined previously. In fact, as will be shown later, $N(\mathbf{r},\omega)$ is identical with the local density of states if the system considered is an equilibrium system. In a non-equilibrium system, $N(\mathbf{r},\omega)$ comprises only a fraction of the total number of possible modes.

The fields due to a set of discrete fluctuating dipoles can be written in a similar form (see Section **??). $\bar{W}_\omega$ can then be derived by using the fluctuation-dissipation theorem of Eq. (14.23].

14.2.1 Blackbody Radiation

Consider a body that is made of fluctuating point sources. Thermal equilibrium with the radiation field implies that the averaged Poynting vector vanishes at all points $\mathbf{r}$ in space (no heat transport). In this case we can use the fluctuation dissipation theorem Eq. (14.32). In free space, the two terms in Eq. (14.47) turn out to be identical and we obtain [9]

$$\bar{W}_\omega(\mathbf{r},\omega) = \left[\frac{\hbar\omega}{1-\mathrm{e}^{-\hbar\omega/kT}}\right]\frac{\omega}{\pi\,c^2}\sum_j \mathrm{Im}\Big\{[\overleftrightarrow{\mathbf{G}}\,(\mathbf{r},\mathbf{r},\omega)]_{jj}\Big\} \quad \text{(equilibrium)} \tag{14.49}$$

Remember, that the total energy is given by integration over positive and negative frequencies. Let us replace the term in brackets by an antisymmetric part and a symmetric part as

$$\frac{\hbar\omega}{2} \;+\; \left[\frac{\hbar\omega}{2} \;+\; \frac{\hbar\omega}{\mathrm{e}^{\hbar\omega/kT}\;-\;1}\right] . \tag{14.50}$$

Considering that $\mathrm{Im}\{\overleftrightarrow{\mathbf{G}}\}$ is an odd function of ω, we can drop the first term in the above expression because its contribution cancels when integrating over positive and negative frequencies. The remaining integral can be written over positive frequencies only as

$$\bar{W} \;=\; \int_0^\infty \bar{W}^+_\omega(\omega)\,d\omega \;=\; \int_0^\infty \bar{w}(\omega,T)\,N(\mathbf{r},\omega)\,d\omega \;\;, \tag{14.51}$$

where

$$\begin{aligned}
\bar{w}(\omega,T) \;&=\; \left[\frac{\hbar\omega}{2} \;+\; \frac{\hbar\omega}{\mathrm{e}^{\hbar\omega/kT}\;-\;1}\right] \\
N(\mathbf{r},\omega) \;&=\; \frac{2\omega}{\pi\,c^2}\sum_j \mathrm{Im}\Big\{[\overleftrightarrow{\mathbf{G}}\,(\mathbf{r},\mathbf{r},\omega)]_{jj}\Big\} \;=\; \frac{2\omega}{\pi\,c^2}\,\mathrm{Im}\Big\{Tr[\overleftrightarrow{\mathbf{G}}\,(\mathbf{r},\mathbf{r},\omega)]\Big\} \;.
\end{aligned}$$

$N(\mathbf{r},\omega)$ is identical with the local density of states [c.f. Eq. (**??**)] and $\bar{w}(\omega,T)$ corresponds to the average energy of a quantum oscillator. $\bar{W}^+_\omega(\omega)$ is the spectral energy density defined over positive frequencies only.

By expanding the exponential term $\exp[ikr]$ in $\overleftrightarrow{\mathbf{G}}$ into a series, it has been shown in Section **??** that $\mathrm{Im}\{\overleftrightarrow{\mathbf{G}}\}$ is *not singular* at its origin. Using the free-space Green's function we obtain $\mathrm{Im}\{[\overleftrightarrow{\mathbf{G}}\,(\mathbf{r},\mathbf{r},\omega)]_{jj}\}=\omega/(6\pi c)$ and Eq. (14.51) becomes

$$\boxed{\bar{W}^+_\omega(\omega) \;=\; \left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\frac{\omega^2}{\pi^2\,c^3}} \;. \tag{14.52}$$

This is the celebrated *Planck's blackbody radiation* formula which renders the electromagnetic energy per unit volume in the frequency range $[\omega\,..\,\omega+d\omega]$. It is strictly only valid for an equilibrium system.

14.2.2 Coherence, Spectral Shifts and Heat Transfer

Thermal equilibrium between matter and the radiation field is practically never encountered. Therefore, the spectral energy density has to be calculated by Eq. (14.47) and the local density of states N becomes position dependent. Shchegrov *et al.* calculated $N(\mathbf{r},\omega)$ near a planar material surface [4] and found that it strongly depends on the distance to the surface. Fig. 14.5 shows the spectral energy density at $T=300K$ above a SiC half-space. At large distances from the surface (Fig. 14.5a), the spectrum looks like a blackbody spectrum multiplied with the SiC emissivity. The latter is responsible for the dip in the spectrum. The emitted radiation is incoherent with a typical coherence length of $\approx \lambda/2$ (Lambertian source). At distances considerably smaller than λ, the spectrum is dominated by a single peak (Fig. 14.5c) which originates from a surface mode (surface phonon polariton). The narrow line width of the peak leads to increased coherence and thus to almost monochromatic fields. The sequence of images clearly indicates that the spectrum changes qualitatively on propagation.

The observed increase of $\bar{W}_\omega$ near material surfaces has implications on *radiative heat transfer*. Radiative heat transfer will occur between two bodies which are kept at different temperatures. However, even a single body in free space will loose its thermal energy by continuous radiation. Muket *et al.* show that the radiative heat-transfer between two separated bodies can be increased by several orders of magnitude as the spacing between the bodies is decreased [10]. This increase originates from the interaction of surface waves localized near the interfaces. This interaction gives rise to heat-transfer limited to a narrow spectral window.

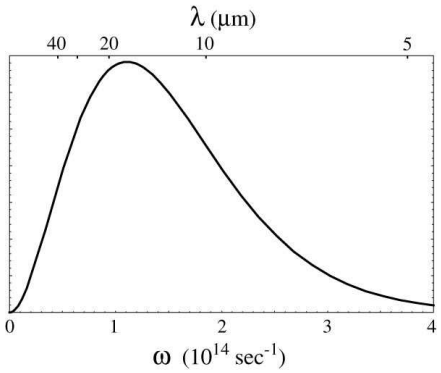

Figure 14.4: Blackbody radiation spectrum $\bar{W}^+_\omega$ for $T=300K$. Equilibrium conditions require that the net Poynting vector vanishes everywhere.

Thermal near-fields not only affect the spectral energy density of the emitted radiation but also their *spatial coherence*. A measure for spatial coherence is given by the electric-field cross-spectral density tensor W_{jk} defined as

$$W_{jk}(\mathbf{r}_1, \mathbf{r}_2, \omega)\, \delta(\omega - \omega') \;=\; \left\langle \delta\hat{E}_j(\mathbf{r}_1, \omega)\, \delta\hat{E}_k^*(\mathbf{r}_2, \omega') \right\rangle . \tag{14.53}$$

Carminati and Greffet have evaluated W_{jk} near surfaces of different materials [11]. They find that an opaque material not supporting a surface mode (e.g. tungsten) can have a spatial coherence length much smaller than the well-known $\lambda/2$ coherence length of blackbody radiation. The coherence length can be arbitrarily small, only limited by non-local effects close to the material surface. On the other hand, near material surfaces supporting surface modes (e.g. silver) the correlation length can reach several tenths of λ.

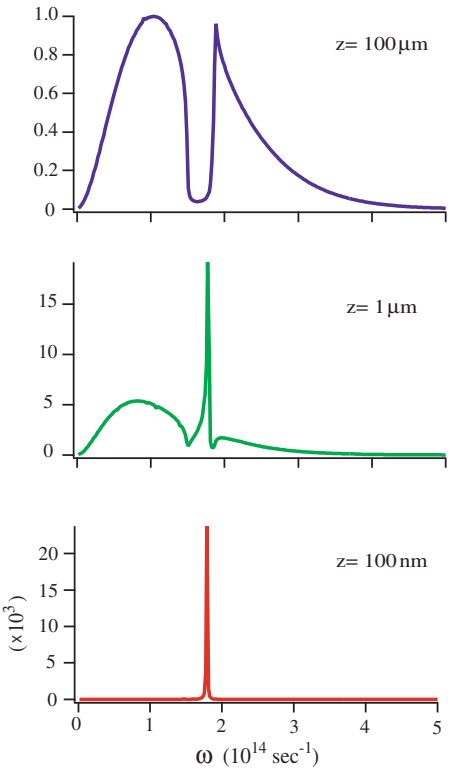

Figure 14.5: Spectra of thermal emission from a semi-infinite sample of SiC at $T = 300K$ evaluated at three different heights z above the surface. From Ref. [4].

14.3 Fluctuation induced forces

Fluctuating charges in a neutral body give rise to fluctuating electromagnetic fields which interact with the charges in other bodies. As a consequence, electromagnetic fields mediate between the charge fluctuations in separate bodies. The resulting charge correlations give rise to an electromagnetic force that is referred to as *dispersion* force. For short distances between the two bodies, the force is called *van der Waals* force, whereas at larger separations it is designated as the *Casimir* force. Although these forces are small on macroscopic scales they cannot be ignored on the scales of nanostructures. For example, two parallel conducting plates with areas $1\mu m^2$ placed $5nm$ apart will experience an attractive force of $\approx 2nN$. This force is sufficient to squash a biomolecule! Dispersion forces are also responsible for weak molecular binding and for adhesion of particles to interfaces. For example, Geckos climb even the most slippery surfaces with ease and hang from glass using a single toe. The secret behind this extraordinary climbing skill lies with millions of tiny keratin hairs on the surface of each foot. Although the dispersion force associated with each hair is miniscule, the millions of hairs collectively produce a powerful adhesive effect. The 'Gecko-effect' is also applied for the design of strongly adhesive tapes. In this section we derive the force acting on a small polarizable particle in an arbitrary environment following the steps of Ref. [12].

To simplify notation we assume that all fluctuations have zero average. This allows us to write $\boldsymbol{\mu}(t) = \delta\boldsymbol{\mu}(t)$ and $\mathbf{E}(t) = \delta\mathbf{E}(t)$. To calculate the force acting on a polarizable particle located at $\mathbf{r} = \mathbf{r}_o$ we use the expression for the gradient force derived in Section **??** [c.f Eq. (**??**)]. However, we have to consider that both field $\mathbf{E}$

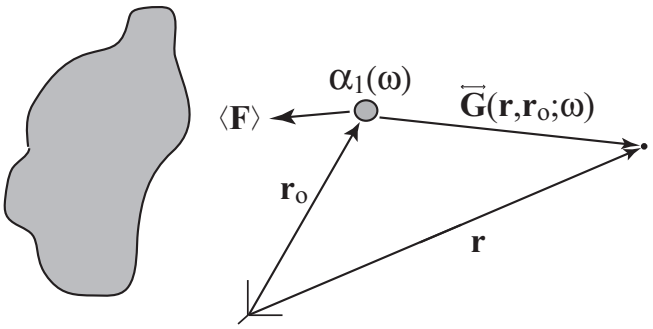

Figure 14.6: Dispersion force acting on a polarizable particle located at $\mathbf{r} = \mathbf{r}_o$. The force originates from correlated charge fluctuations in particle and other bodies in the environment. The latter are accounted for by the Green's function $\overleftrightarrow{\mathbf{G}}$ evaluated at its origin.

and dipole moment $\boldsymbol{\mu}$ have fluctuating and induced parts. Therefore,

$$\langle \mathbf{F}(\mathbf{r}_o) \rangle \;=\; \sum_i \Big[\langle \mu_i^{(in)}(t)\, \nabla E_i^{(fl)}(\mathbf{r}_o,t) \rangle \;+\; \langle \mu_i^{(fl)}(t)\, \nabla E_i^{(in)}(\mathbf{r}_o,t) \rangle \Big] \;, \tag{14.54}$$

where $i = \{x,y,z\}$. The first term describes the (spontaneous and thermal) field fluctuations that correlate with the induced dipole according to

$$\hat{\boldsymbol{\mu}}^{(in)}(\omega) \;=\; \alpha_1(\omega)\, \hat{\mathbf{E}}^{(fl)}(\mathbf{r}_o,\omega) \;, \tag{14.55}$$

where we assumed an isotropic polarizability. For later purposes, we denote the properties of the particle by an index '1'. The second term in Eq. (14.54) originates from the particle's dipole fluctuations and the corresponding induced field according to

$$\hat{\mathbf{E}}^{(in)}(\mathbf{r},\omega) \;=\; \frac{\omega^2}{c^2}\frac{1}{\varepsilon_o}\; \overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}_o;\omega) \cdot \hat{\boldsymbol{\mu}}^{(fl)}(\omega) \;. \tag{14.56}$$

Here, $\overleftrightarrow{\mathbf{G}}$ is the Green's function of the reference system and $\mathbf{r}$ denotes an arbitrary field point as visualized in Fig. 14.6. Correlations between fluctuating field and fluctuating dipole are zero because they originate from different physical systems. Likewise, there are no correlations between the induced quantities.

After expressing $\boldsymbol{\mu}$ and $\mathbf{E}$ in Eq. (14.54) by their Fourier transforms and making use of the fact that $\mathbf{E}(t) = \mathbf{E}^*(t)$ we obtain

$$\langle \mathbf{F}(\mathbf{r}_o) \rangle \;=\; \sum_i \iint_{-\infty}^{\infty} \big\langle \hat{\mu}_i^{(in)}(\omega)\; \nabla \hat{E}_i^{*\,(fl)}(\mathbf{r}_o,\omega') \big\rangle\; \mathrm{e}^{i(\omega'-\omega)t}\, d\omega' d\omega \;+ \tag{14.57}$$

$$\sum_i \iint_{-\infty}^{\infty} \big\langle \hat{\mu}_i^{(fl)}(\omega)\; \nabla \hat{E}_i^{*\,(in)}(\mathbf{r}_o,\omega') \big\rangle\; \mathrm{e}^{i(\omega'-\omega)t}\, d\omega' d\omega \;. \tag{14.58}$$

Introducing the linear relationships 14.55 and 14.56 and arranging terms allows us to express the first term as a function of $\hat{\mathbf{E}}^{(fl)}$ and the second term as a function of $\hat{\boldsymbol{\mu}}^{(fl)}$ as

$$\langle \mathbf{F}(\mathbf{r}_o) \rangle \;=\; \sum_i \iint_{-\infty}^{\infty} \alpha_1(\omega)\, \nabla_2 \big\langle \hat{E}_i^{*\,(fl)}(\mathbf{r}_o,\omega)\, \hat{E}_i^{*\,(fl)}(\mathbf{r}_o,\omega') \big\rangle\; \mathrm{e}^{i(\omega'-\omega)t}\, d\omega' d\omega \tag{14.59}$$

$$+ \sum_{i,j} \iint_{-\infty}^{\infty} \frac{\omega'^2}{c^2}\frac{1}{\varepsilon_o}\, \nabla_1 G_{ij}^*(\mathbf{r}_o,\mathbf{r}_o;\omega') \big\langle \hat{\mu}_i^{(fl)}(\omega)\; \hat{\mu}_j^{*\,(fl)}(\omega') \big\rangle\; \mathrm{e}^{i(\omega'-\omega)t}\, d\omega' d\omega \;,$$

where ∇_n specifies that the gradient has to be taken with respect to the n-th spatial variable in the argument. Using the fluctuation-dissipation theorems for dipole and field [Eqs. (14.19) and (14.32)] and the fact that

$$\nabla_1\, \overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}_o;\omega) \;=\; \nabla_2\, \overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}_o;\omega) \tag{14.60}$$

allows us to write the force in the compact form

$$\boxed{\langle \mathbf{F}(\mathbf{r}_o)\rangle = \sum_i \int_{-\infty}^{\infty} \frac{\omega}{\pi c^2 \varepsilon_o}\left[\frac{\hbar\omega}{1-\mathrm{e}^{-\hbar\omega/kT}}\right] \mathrm{Im}\Big\{\alpha_1(\omega)\,\nabla_1 G_{ii}(\mathbf{r}_o,\mathbf{r}_o;\omega)\Big\}\, d\omega} \quad (14.61)$$

Notice, that the force is determined by the properties of the environment which is encoded in the Green's function $\overleftrightarrow{\mathbf{G}}$. The force vanishes in the absence of any objects, i.e. when $\overleftrightarrow{\mathbf{G}}$ equals the free space Green's function. Eq. (14.61) allows us to calculate the force acting on a small polarizable particle in an arbitrary environment. The equation is valid for an isotropic particle but it can be generalized to account for anisotropic polarizabilities such as for molecules with fixed transition dipole moments.

14.3.1 The Casimir-Polder potential

In this section we derive the force acting on a particle with polarizability α_1 due to another particle with polarizability α_2. As indicated in Fig. 14.7, the two particles are separated by a distance R. For short distances, the force varies as R^{-7} whereas for larger distances the force assumes a R^{-8} dependence. The stronger distance dependence at large distances is counter intuitive since the decay of electromagnetic fields becomes weaker when going from the near-field to the farfield. It will be shown that for temperatures $T = 0$, the force at all distances can be deduced from a single potential $U(R)$, called the *Casimir-Polder potential*. Finite temperatures affect the force only marginally [12] and hence we will restrict the analysis to the case $T = 0$.

The force in Eq. (14.61) is defined by the Green's function $\overleftrightarrow{\mathbf{G}}$. Therefore, let us derive the Green's function accounting for the presence of a polarizable particle with

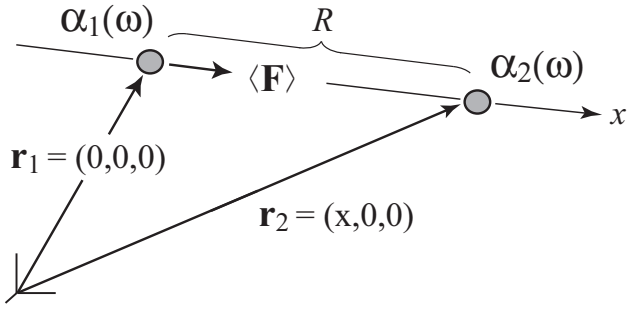

Figure 14.7: Definition of coordinates for the calculation of the dispersion force between two polarizable particles.

polarizability α_2 centered at $\mathbf{r}_2$. The field $\mathbf{E}$ at $\mathbf{r}$ due to a dipole at $\mathbf{r}_1$ can be expressed as

$$\hat{\mathbf{E}}(\mathbf{r},\omega) \;=\; \frac{\omega^2}{c^2}\frac{1}{\varepsilon_o}\;\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r},\mathbf{r}_1;\omega)\,\hat{\boldsymbol{\mu}}_1(\omega) \;+\; \hat{\mathbf{E}}_s(\mathbf{r},\omega)\;, \tag{14.62}$$

with $\overleftrightarrow{\mathbf{G}}^{o}$ denoting the free-space Green's dyadic. The scattered field $\hat{\mathbf{E}}_s$ originates from the particle at $\mathbf{r}_2$ and is determined as

$$\begin{aligned}\hat{\mathbf{E}}_s(\mathbf{r},\omega) \;&=\; \frac{\omega^2}{c^2}\frac{1}{\varepsilon_o}\;\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r},\mathbf{r}_2;\omega)\,\hat{\boldsymbol{\mu}}_2(\omega) \\ &=\; \frac{\omega^2}{c^2}\frac{1}{\varepsilon_o}\left[\frac{\omega^2}{c^2}\frac{1}{\varepsilon_o}\;\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r},\mathbf{r}_2;\omega)\;\alpha_2(\omega)\;\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r}_2,\mathbf{r}_1;\omega)\right]\hat{\boldsymbol{\mu}}_1(\omega)\;.\end{aligned} \tag{14.63}$$

Combining Eq. (14.62) and (14.63) allows us to identify the Green's function of the system of 'free-space + particle at $\mathbf{r}_2$' as

$$\overleftrightarrow{\mathbf{G}}\,(\mathbf{r},\mathbf{r}_1;\omega) \;=\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r},\mathbf{r}_1;\omega) \;+\; \frac{\omega^2}{c^2}\frac{1}{\varepsilon_o}\;\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r},\mathbf{r}_2;\omega)\;\alpha_2(\omega)\;\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r}_2,\mathbf{r}_1;\omega)\;. \tag{14.64}$$

The gradient of $\overleftrightarrow{\mathbf{G}}$ evaluated at its origin $\mathbf{r} = \mathbf{r}_1$ is:

$$\nabla_1\,\overleftrightarrow{\mathbf{G}}\,(\mathbf{r}_1,\mathbf{r}_1;\omega) \;=\; \frac{\omega^2}{c^2}\frac{1}{\varepsilon_o}\,\alpha_2(\omega)\,\left[\nabla_1\,\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r}_1,\mathbf{r}_2;\omega)\right]\,\overleftrightarrow{\mathbf{G}}^{o}(\mathbf{r}_2,\mathbf{r}_1;\omega)\;. \tag{14.65}$$

Let us choose the coordinates as $\mathbf{r}_1 = \mathbf{0}$ and $\mathbf{r}_2 = (x,0,0) = x\,\mathbf{n}_x$. We then obtain for the sum of the diagonal elements of $\nabla\overleftrightarrow{\mathbf{G}}$

$$\sum_i \nabla_1 G_{ii}(\mathbf{r}_1,\mathbf{r}_1;\omega) \;=\; \frac{\omega^2}{c^2}\frac{1}{\varepsilon_o}\,\alpha_2(\omega)\;\sum_i\left[\frac{\partial}{\partial x}G^o_{ii}(i,0;\omega)\right]G^o_{ii}(i,0;\omega)\;, \tag{14.66}$$

where we made use of the properties of the free-space Green's function $\overleftrightarrow{\mathbf{G}}^{o}$. Using the explicit form of $\overleftrightarrow{\mathbf{G}}^{o}$ in the above expression gives

$$\begin{aligned}\sum_i \nabla_1\,G_{ii}(\mathbf{r}_1,\mathbf{r}_1;\omega) \;&=\; \frac{c^2}{\omega^2}\frac{1}{\varepsilon_o}\,\frac{\exp(2ix\omega/c)}{8\pi^2x^7}\,\alpha_2(\omega)\,\Big[-9+18i\left(\frac{\omega}{c}x\right)\;+ \\ &\qquad 16\left(\frac{\omega}{c}x\right)^2\;-\;8i\left(\frac{\omega}{c}x\right)^3\;-\;3\left(\frac{\omega}{c}x\right)^4\;+\;i\left(\frac{\omega}{c}x\right)^5\Big]\;\mathbf{n}_x \\ &=\; \sum_i \nabla_1\,G_{ii}(x;\omega)\;.\end{aligned} \tag{14.67}$$

We now introduce this Green's function into the force formula 14.61 which, for $T = 0$, reads as

$$\left\langle\mathbf{F}(x)\right\rangle = \frac{\hbar}{\pi c^2\varepsilon_o}\int_0^\infty \omega^2\;\mathrm{Im}\left\{\alpha_1(\omega)\;\sum_i\nabla_1 G_{ii}(x;\omega)\right\}\;d\omega\;. \tag{14.68}$$

Here, we made use of the fact that contributions with negative frequencies vanish [c.f. Eq. (14.20)].

It is straightforward to show that $\nabla \times \langle \mathbf{F} \rangle = 0$ and hence the force is conservative. Therefore, we can derive the force from a potential U by integration over the variable x. We obtain

$$U = -\int \langle F(x) \rangle \, dx \;=\; \frac{\hbar}{16\pi^3 \varepsilon_o^2 x^6} \, \mathrm{Im} \int_0^\infty \alpha_1(\omega) \, \alpha_2(\omega) \, \mathrm{e}^{2ix\omega/c} \times \left[-3 + 6i \left(\frac{\omega}{c} x\right) + 5 \left(\frac{\omega}{c} x\right)^2 - 2i \left(\frac{\omega}{c} x\right)^3 - \left(\frac{\omega}{c} x\right)^4 \right] d\omega \; . \tag{14.69}$$

We now substitute the integration variable as $\tilde{\omega} = \omega c$ and replace the interparticle distance by R. We then realize that the integrand is analytic in the upper half-space of the integration variable and that the integrand goes to zero as $\tilde{\omega} \to \infty$. Therefore, we can integrate along the imaginary axis using

$$\int_0^\infty f(\tilde{\omega}) \, d\tilde{\omega} \;=\; i \int_0^\infty f(i\eta) \, d\eta \; . \tag{14.70}$$

Combining these mathematical tricks we obtain for the interparticle potential

$$\boxed{U = -\frac{\hbar c}{16\pi^3 \varepsilon_o^2 \, R^6} \int_0^\infty \alpha_1(ic\eta) \, \alpha_2(ic\eta) \, \mathrm{e}^{-2\eta R} \left[3 + 6\eta R + 5(\eta R)^2 + 2(\eta R)^3 + (\eta R)^4 \right] d\eta} \tag{14.71}$$

We made use of the fact that $\alpha_i(\Omega)$ is purely real on the imaginary axis $\Omega = i\eta$. Eq. (14.71) is the celebrated *Casimir-Polder* potential valid for any interparticle separations R. Our result agrees with rigorous calculations based on quantum electrodynamics using fourth-order perturbation theory [13]. Our model allows us to incorporate higher order corrections by simply adding additional interaction terms to the Green's function $\overleftrightarrow{\mathbf{G}}$ in Eq. (14.64). The force can be retrieved from the potential using $\langle \mathbf{F} \rangle = -\nabla U$.

It is interesting to evaluate the potential for the limiting cases of large and small interparticle distances. For short distances we retain only the first term in the bracket, set $\exp(2\eta R) = 1$, and obtain

$$\boxed{U(R \to 0) = -\frac{6\hbar}{32\pi^3 \varepsilon_o^2} \, \frac{1}{R^6} \int_0^\infty \alpha_1(i\eta) \, \alpha_2(i\eta) \, d\eta} \; . \tag{14.72}$$

This is the *van der Waals* force valid for short interparticle distances R. The force depends on the dispersive properties of the particle polarizabilities and scales with the inverse sixth power of the particle separation R.

To obtain the limit for large R, we make the substitution $u = \eta R$ in Eq. (14.71) which leads to the following expression for the interparticle potential

$$U = -\frac{\hbar c}{16\pi^3 \varepsilon_o^2 \, R^7} \int_0^\infty \alpha_1(icu/R)\, \alpha_2(icu/R)\, \mathrm{e}^{-2u} \big[3 + 6u + 5u^2 + 2u^3 + u^4\big] du. \quad (14.73)$$

Then, in the large distance limit ($R \to \infty$), one can replace the polarizabilities by their static values $\alpha_i(0)$. After moving the polarizabilities out of the integral one obtains

$$U(R \to \infty) = -\frac{\hbar c}{16\pi^3 \varepsilon_o^2} \, \frac{\alpha_1(0)\, \alpha_2(0)}{R^7} \int_0^\infty \mathrm{e}^{-2u} \left[3 + 6u + 5u^2 + 2u^3 + u^4\right] du \; . \quad (14.74)$$

Finally, using the equality

$$\int_0^\infty u^n \, \mathrm{e}^{-2u} \, du = \frac{n!}{2^{n+1}} \quad \forall \; n \geq 0 \; , \quad (14.75)$$

one can analytically perform the integration in Eq. (14.74). We then obtain the Casimir-Polder interparticle potential in the limit of large distances as

$$\boxed{U(R \to \infty) = -\frac{23\hbar c}{64\pi^3 \varepsilon_o^2} \, \frac{\alpha_1(0)\, \alpha_2(0)}{R^7}} \; . \quad (14.76)$$

This result is a pure manifestation of vacuum fluctuations and it is referred to as the Casimir potential which was first derived in 1948 by Hendrick Casimir [14]. It is remarkable that the potential scales with the inverse seventh power of the interparticle distance R. Thus, for large distances the force decays more rapidly than for short distances. This behavior is opposite to the case of electromagnetic energydensity which shows the fastest decay (R^{-6}) close to the sources. The Casimir potential depends only on the static ($\omega = 0$) polarizabilities of the particles and hence it doesn't matter what their spectral properties are. Notice, that in deriving the Casimir-Polder potential we considered only the gradient force and neglected the influence of the scattering force. The scattering force is non-conservative and it must be zero if the particle(s) remain in equilibrium with the vacuum field.

It has to be emphasized, that the Casimir-Polder potential originates solely from zero-point fluctuations and it does not account for thermal fluctuations. At room temperature, thermally induced forces are usually more than one order of magnitude weaker than the forces associated with vacuum fluctuations [12].

14.3.2 Electromagnetic Friction

Electromagnetic interactions between two charge-neutral objects gives not only rise to conservative dispersion forces but also to a *non-conservative* friction force if the two objects are in relative motion to each other. This friction force is associated only with thermal fluctuations and it brings the motion of an object ultimately to rest. Although this force is small, it has direct consequences for the development of nano-electro-mechanical systems (NEMS) and for various proposals in the field of quantum information. Electromagnetic friction will lead to increased decoherence in miniaturized particle traps such as ion-traps and atom chips and limit the Q-factor of mechanical resonances.

Let us consider a small, charge-neutral particle such as an atom, molecule, cluster, or a macroscopic nanoscale structure that is small compared to all relevant wavelengths λ. In this limit, the particle is represented by the polarizability $\alpha(\omega)$. The particle is placed in an arbitrary environment characterized by the Green's function $\overleftrightarrow{\mathbf{G}}$ and we assume that its motion of the center of mass coordinate $x(t)$ is governed by the classical Langevin equation

$$m\frac{d^2}{dt^2}x\left(t\right)+\int_{-\infty}^{t}\gamma\left(t-t'\right)\frac{d}{dt'}x\left(t'\right)\,dt'+mw_o^2x\left(t\right)\;=\;F_x(t). \tag{14.77}$$

Here, m is the mass of the particle, $\gamma(t)$ is the damping coefficient originating from thermal electromagnetic field fluctuations, w_o is the natural frequency of the oscillating particle and $F_x(t)$ is the stochastic force. Note, that the restoring force $mw_o^2x\left(t\right)$ is only added for generality and does not influence the final result. In thermal equilibrium, $F_x(t)$ is a stationary stochastic process with zero ensemble average. The spectral force spectrum $S_{\mathrm{F}}\left(\omega\right)$ is given by the Wiener-Khintchine theorem as [c.f. Eq. (14.16)]

$$S_{\mathrm{F}}\left(\omega\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle F_x\left(\tau\right)F_x\left(0\right)\right\rangle\,e^{\mathrm{i}\omega\tau}d\tau, \tag{14.78}$$

where ω is the angular frequency. Furthermore, at thermal equilibrium S_{F} is related to the friction coefficient by the fluctuation-dissipation theorem. Because the motion of the macroscopic particle is classical we consider the classical limit, i.e.

$$kT\,\hat{\gamma}\left(\omega\right)\;=\;\pi\,S_{\mathrm{F}}\left(\omega\right). \tag{14.79}$$

with $\hat{\gamma}\left(\omega\right)$ being the Fourier transform of $\gamma\left(t\right)$ defined *only* for $t>0$.

In Eq. (14.77), we assumed a general friction force term whose magnitude at time t depends on the particle's velocity at earlier times. We now consider that the interaction time of the thermal bath with the particle is fast compared with the particle's dynamics, thus the change of the velocity of the particle during the interaction time is very small. In this Markovian approximation friction has no memory and thus

$$F_{\mathrm{friction}}(t)\;=\;-\gamma_o\frac{d}{dt}x(t)\;,\qquad\gamma_o=\int_0^{\infty}\gamma(t)dt. \tag{14.80}$$

Evaluating Eq. (14.79) at $\omega = 0$ and using Eq. (14.80) we find that the damping constant is related to the force spectrum as

$$kT\gamma_o \quad = \quad \pi\, S_{\rm F}(\omega = 0). \tag{14.81}$$

This is the final expression that relates the linear-velocity damping coefficient to the force spectrum. To calculate γ_o, we need to solve for the force spectrum which, in turn, is defined by the electromagnetic fields due to fluctuating currents in the environment and the fluctuating dipole [c.f. Eq. (14.54)].

Using the Wiener-Khintchine theorem (14.78), the Fourier transform of the dipole force (14.54), and the stationarity of the fluctuations we obtain

$$\begin{aligned}\left\langle \hat{F}_x^*(\omega')\hat{F}_x(\omega)\right\rangle \quad &= \quad S_{\rm F}\,(\omega)\,\delta(\omega-\omega') \\ &= \sum_{i,j=1}^{3}\left\langle \left[\left(\hat{\mu}_j^{*\,(fl)}(\omega')+\hat{\mu}_j^{*\,(in)}(\omega')\right)\otimes\left(\frac{\partial}{\partial x}\hat{E}_j^{*\,(fl)}(\omega')+\frac{\partial}{\partial x}\hat{E}_j^{*\,(in)}(\omega')\right)\right]\right. \\ &\quad \left.\times\left[\left(\hat{\mu}_i^{(fl)}(\omega)+\hat{\mu}_i^{(in)}(\omega)\right)\otimes\left(\frac{\partial}{\partial x}\hat{E}_i^{(fl)}(\omega)+\frac{\partial}{\partial x}\tilde{E}_i^{(in)}(\omega)\right)\right]\right\rangle, \end{aligned} \tag{14.82}$$

where $\otimes$ denotes convolution. Each of the additive terms in $\langle\tilde{F}_x^*(\omega')\tilde{F}_x(\omega)\rangle$ is a fourth-order frequency-domain correlation function. Because the fluctuation-dissipation theorem involves second-order correlations and not fourth-order correlations it is not possible to find a solution using near-equilibrium statistical mechanics. However, there is a way out: thermal fluctuating fields can be thought of as arising from the superposition of a large number of radiating oscillators with a broad-band spectrum. Consequently, the central-limit theorem applies. The same is true for the dipole fluctuations because of their broad thermal spectrum. Stochastic processes with Gaussian statistics have the property that a fourth-order correlation function can be expressed by a sum of pair-products of second-order correlation functions. Thus, Eq. (14.82) can be calculated by knowing the second-order correlations of the thermal electromagnetic fields and the electric dipole fluctuations. At *thermal equilibrium*, these correlation functions are given by the fluctuation-dissipation theorems Eq. (14.19) and (14.32). Thus, we have all the ingredients to calculate the damping coefficient γ_o in Eq. (14.81). We replace the induced terms in Eq. (14.82) by the fluctuating terms using the linear relationships Eqs. (14.55) and (14.56). Then we introduce the fluctuation-dissipation theorems Eqs. (14.19) and (14.32). Finally, we make use of Eq. 14.81 through which we find the spectrum of the damping constant γ_o. The four additive terms in Eq. (14.82) lead to four additive damping constants of which two are negligibly small.

It can be shown that friction disappears as $T \to 0$ which indicates that friction is only associated with thermal fluctuations and not with quantum zero-point

fluctuations. In fact, this result is also implied by the requirement that zero-point fluctuations are invariant under the Lorentz transformation [15]. Furthermore, another remarkable result is that friction is even present in empty space as long as the temperature is finite. Thus, an object moving in empty space comes ultimately to rest. In the free-space limit we obtain

$$\gamma_o = \frac{\hbar^2}{18\pi^3 c^8 \varepsilon_o^2 kT} \int_0^\infty |\alpha(\omega)|^2 \, \omega^8 \eta(\omega, T) \, d\omega + \frac{\hbar^2}{3\pi^2 c^5 \varepsilon_o kT} \int_0^\infty \mathrm{Im}[\alpha(\omega)] \omega^5 \eta(\omega, \mathrm{T}) \, \mathrm{d}\omega \, , \tag{14.83}$$

where

$$\eta(\omega, T) \equiv \left[1/(\mathrm{e}^{\hbar\omega/\mathrm{kT}} - 1)\right] \left[1 + 1/(\mathrm{e}^{\hbar\omega/\mathrm{kT}} - 1)\right] \, . \tag{14.84}$$

The first term in Eq. (14.83) is consistent with the result by Boyer [15] whereas the second term was independently derived in Refs. [17, 16].

In Ref. [17], electromagnetic friction has been analyzed for the special case of a polarizable spherical particle placed near a semi-infinite half-space (substrate) with a complex dielectric constant $\varepsilon_2(\omega)$. Similar studies have been presented in Refs. [18, 19]. It is assumed that the particle is moving parallel to the surface (x-direction) at a vertical height of z_o (see Fig. 14.9). These studies not only revealed a steep distance dependence of the damping constant but also a strong dependence on the material properties of particle and substrate. As an example, Fig. 14.9 shows the normalized spectral density of the damping coefficient as a function of angular frequency ω and for temperatures $T = 3\,\mathrm{K}, 30\,\mathrm{K}, 300\,\mathrm{K}$. The damping coefficient γ_o is obtained by integrating the curves over all frequencies, i.e. the area under the spectral curves. The figure shows the results for two different material configurations: (a) sample and particle made of silver and (b) sample and particle made of glass. It is evident

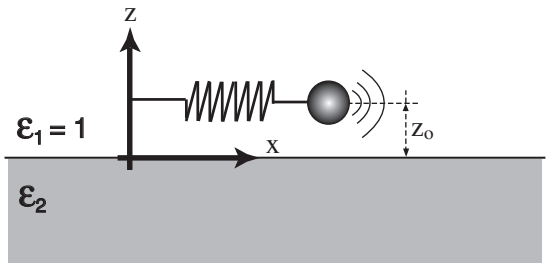

Figure 14.8: A particle in vacuum moves parallel to a planar substrate with a dielectric function $\varepsilon_2(\omega)$.

that the spectral range of γ is much shorter in Fig. 14.9a than in Fig. 14.9b. On the other hand, the amplitude is much lower. Thus, while a dielectric system has a much shorter bandwidth it leads to a much stronger damping amplitude as compared with a metallic system. Numerical integration renders the values listed in Table 14.1. The important outcome is that the magnitude of the damping coefficient is largely determined by the material of the semi-infinite substrate (c.f. Ref. [17]). The particle properties have only a minor effect. Also important is the finding that only thermal fluctuations in the low-frequency range of $0\,..\,100$ Hz are significant.

At first glance, the difference of 19 orders of magnitude between the glass and the silver substrate is very surprising. However, it can be explained by the following qualitative physical picture. The fluctuating currents in particle and substrate generate a fluctuating electromagnetic field. This field polarizes the particle and induces an electric dipole with a corresponding image dipole beneath the surface of the substrate. The motion of the particle gives rise to motion of the image dipole. The Joule losses associated with the motion of the image dipole become larger with increasing resistivity of the substrate. As a consequence, the damping coefficient increases, too. Physically, more work is needed to move the induced dipole beneath the surface as the resistivity increases and, consequently, the damping coefficient becomes larger. In the limit of a perfect dielectric the induced dipole cannot be displaced and damping becomes infinitely strong. On one hand, it is surprising to find this result for a perfect (lossless) dielectric since there is no intrinsic dissipation. On the other hand, a lossless dielectric does not exist from the point of view of causality (Kramers-Kronig relations) and the fluctuation-dissipation theorem (fluctuations imply dissipation). Neverthe-

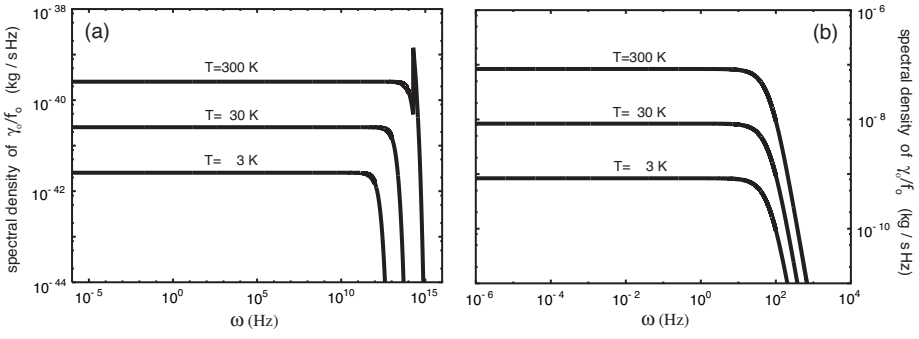

Figure 14.9: Normalized spectral density of the damping coefficient as a function of the angular frequency ω for temperatures $T = 3\,\mathrm{K}$, $30\,\mathrm{K}$, $300\,\mathrm{K}$. $f_o = a^3/z_o^5$, where a (particle radius) and z_o (tip-sample distance) are defined in nanometers. (a) Substrate and particle made of a metal (Ag); (b) substrate and particle made of a dielectric (SiO_2). From Ref. [17].

Substrate	Particle	γ_o/f_o (kg/s)		
		3 K	30 K	300 K
Silver	Silver	3.28×10^{-30}	3.28×10^{-28}	1.01×10^{-25}
Glass	Glass	4.65×10^{-08}	4.65×10^{-07}	4.65×10^{-06}

Table 14.1: Normalized damping coefficient γ_o/f_o for a spherical particle calculated for several temperatures ($T = 3\,\mathrm{K}, 30\mathrm{K}, 300\,\mathrm{K}$). $f_o = a^3/z_o^5$, with units of nanometers.

less, in the limit $T \to 0$, the damping coefficient vanishes even for a perfect dielectric. Notice that because γ_o is much weaker for metals, local friction measurements render metals transparent and reveal buried dielectric structures. This property can be used for subsurface imaging in metals and for the localization of defects.

Problems

Problem 14.1 Derive Eq. (14.10) by using the series expansion of Eq. (14.9) for the distribution function.

Problem 14.2 Eq. (14.47) describes the spectral energy density W_ω as a function of the dielectric constant $\varepsilon(\mathbf{r},\omega)$. Derive a similar equation for a system of N particles with coordinates $\mathbf{r}_n$ and polarizabilities $\alpha_n(\omega)$. Hint: Use the fluctuation-dissipation theorem in Eq. (14.19).

Problem 14.3 Determine the spectral energy density W_ω due to fluctuating sources inside a single dipolar particle (diameter $\ll \lambda$) with polarizability α. Show that the electric and magnetic energy densities are identical and that there is *no* near-field contribution. Hint: Use the Green's functions defined in Eqs. (**??**) and (**??**).

Problem 14.4 The polarizability of an aluminum cluster can be approximated by the Clausius-Mossotti formula

$$\alpha(\omega) \;=\; 3\varepsilon_o V_o \frac{\varepsilon(\omega)-1}{\varepsilon(\omega)+2}\,, \tag{14.85}$$

where V_o is the volume of the cluster and ε is the dielectric constant of aluminum. The latter is described by the Drude model as

$$\varepsilon(\omega) \;=\; 1-\frac{\omega_p^2}{\omega^2+i\gamma\omega}\,, \tag{14.86}$$

with ω_p and γ being the plasma frequency and the damping constant, respectively. A good approximation is obtained using the values $\hbar\omega_p = 15.565eV$ and $\hbar\gamma = 0.608eV$. Calculate the mean-square of the fluctuating dipole moment in the frequency range $\omega \,..\, \omega{+}d\omega$ and plot this quantity as a function of frequency for temperatures $kT \ll \hbar\omega_p$. Determine the total radiated power.

Problem 14.5 Derive the force formula (14.61) starting with equation (14.54) and following the steps outlined in section 14.3.

Problem 14.6 The polarizability of a Helium atom can be approximated by a single Lorentzian function as

$$\alpha(\omega) \;=\; \frac{(e^2/m_e)\, f_o}{\omega_o^2-\omega^2-i\omega\gamma_o}\,,$$

where the resonance frequency ω_o accounts for all ${}^1S \rightarrow {}^1P^o$ transitions. The oscillator strength is related to the static polarizability as $f_o = \alpha(0)\,\omega_o^2\,(m_e/e^2)$, and γ_o is the effective linewidth.

1. Derive $\alpha(i\eta)$ and show that it is real. Make use of $\gamma_o \ll \omega_o$.

2. The van der Waals potential between two He atoms can be represented as $U_v = -C_6/R^6$. Calculate the coefficient C_6 and express it in terms of $\alpha(0)$ and $\hbar\omega_o$. The resulting expression is known as *London's empirical formula*.

$$\text{Hint}: \quad \int_\infty^\infty \frac{1}{(A^2+x^2)^2}\, dx = \frac{\pi}{2A^3} \tag{14.87}$$

3. Determine the distance R_o for which U_v is equal to the Casimir potential (U_c). Use $\lambda_o = 2\pi\, c/\omega_o \approx 58nm$.

4. The static polarizability is $\alpha(0) = 2.280\,10^{-41}\, Cb\, m^2\,/\,V$ and ω_o is given by $\lambda_o \approx 58nm$. Plot the Casimir-Polder potential (U_{cp}) as a function of R. Include the curves for U_v and U_c and discuss the validity of these approximations. Provide the value of U_{cp} at the distance R_o.

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