Diffraction by a Small Circular Aperture

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I. OVERVIEW OF DIFFRACTION THEORY

The first reference to the diffraction phenomena appears in the notes of Leonardo da Vinci and it was widely studied after that by Grimaldi, Newton, Young, Fresnel, and others. However, it was not until 1882 that the theory was put on mathematical grounds by Kirchhoff. The vectorial generalization of the theory was developed by Smythe in 1947. [1]

A rigorous analysis of the theory of diffraction is based on Maxwell's equations and the appropriate boundary conditions, which are used in order to calculate the scattered fields that result from induced currents in the diffracting object.

The basic foundation of scalar diffraction theory is Green's theorem, which is used to express the scalar field inside a closed volume in terms of the values of the field and its normal derivatives on the boundary surface. The general result of scalar diffraction theory is given by:

$$\psi(\vec{r}) = \oint_S \left(\psi(\vec{r}\,')\hat{n}\cdot\nabla' G(\vec{r},\vec{r}\,') - G(\vec{r},\vec{r}\,')\hat{n}\cdot\nabla'\psi(\vec{r}\,')\right) da' \quad (1)$$

where G is the Green's function. The only approximation made in eq. *(1)* is the use of scalar theory.

The Kirchhoff diffraction integral, which is obtained by using the free-space Green's function describing outgoing waves and assuming that the field satisfies the radiation condition, is given by:

$$\psi(\vec{r}) = -\frac{1}{4\pi}\int_{S_I} \frac{\exp(ikR)}{R}\hat{n}\cdot\left[\nabla' + ik\left(1+\frac{i}{kR}\right)\frac{\vec{R}}{R}\right]\psi(\vec{r}\,')da' \quad (2)$$

where S_I indicates integration over the plane of the aperture, which is considered to be infinitely extended.

The main problem with this result is that both the field and its normal derivative need to be known on the plane of the aperture. In order to use this result some approximations, known as Kirchhoff's approximations, need to be made:

- *The field and its normal derivative vanish everywhere on the surface except in the aperture.*
- *Their value in the aperture is equal to the value of the incident field in the absence of the screen.*

The first assumption introduces a mathematical inconsistency in the analysis, since it implies that the field should be zero everywhere. Such inconsistency can be removed by the proper choice of Green's function, such that the Dirchlet ($G_D=0$ on the surface) or Neumann ($\partial G_N/\partial n = 0$ on the surface) boundary conditions are satisfied.

Using the Dirchlet Green's function and the radiation condition, eq. *(1)* reduces to:

$$\psi(\vec{r}) = \frac{k}{2\pi i}\int_{S_I}\frac{\exp(ikR)}{R}\left(1+\frac{i}{kR}\right)\frac{\hat{n}\cdot\vec{R}}{R}\psi(\vec{r}\,')\,da', \quad (3)$$

which is known as the Rayleigh-Sommerfeld diffraction integral. This result is similar to Kirchhoff's integral [eq. *(2)*] but without the normal derivative of the field, so that the inconsistency is removed, making this result self-consistent. [2]

Two different approximations are usually made for the phase factor of eq. *(3)*: a second order approximation of R results in Fresnel diffraction theory, while a first order approximation gives Fraunhofer diffraction theory, in which the diffracted field is proportional to the Fourier transform of the field in the aperture.

The main problem with the theory presented so far is its scalar nature, so it is necessary to generalize to a vectorial analysis. Starting with "generalized" Maxwell's equations[1] in free-space:

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0 \qquad \nabla\cdot\vec{H} = \rho^*/\mu_0$$
$$\nabla\times\vec{E} - i\omega\mu_0\vec{H} = -\vec{J}^* \quad \nabla\times\vec{H} + i\omega\varepsilon_0\vec{E} = \vec{J} \quad (4)$$

where the magnetic charge (ρ^*) and current ($\vec{J}^*$) densities have been introduced for convenience. These quantities have no physical meaning. Using the vectorial analogue of Green's theory, the general result for the diffracted field is obtained as:

$$\vec{E}(\vec{r}) = \int_V\left(i\omega\mu_0\vec{J}G - \vec{J}^*\times\nabla G + \frac{1}{\varepsilon_0}\rho\nabla G\right)dV + \int_{S_I}\left[i\omega\mu_0(\hat{n}\times\vec{H})G + (\hat{n}\times\vec{E})\times\nabla' G + (\hat{n}\cdot\vec{E})\nabla' G\right]da' \quad (5)$$

The currents and charges in the volume V are the source of the electric field, and since a closed volume is being considered, the contributions of such currents and charges from the boundary of the volume (surface currents and charges) must also be taken into account. It is possible to identify the terms in the surface integrals as an electric surface charge (η^*) and current ($\vec{K}^*$) densities and a magnetic surface current density ($\vec{K}$):

$$\vec{K}^* = \hat{n}\times\vec{H},\ \vec{K} = -\hat{n}\times\vec{E},\ \eta^* = \varepsilon_0\hat{n}\cdot\vec{E} \quad (6)$$

An analogous expression for $\vec{H}$ can be obtained by means of the substitutions $\vec{E}\to\vec{H}$ and $\vec{H}\to-\vec{E}$. [3]

The vectorial generalization of eq. *(2)* is given by:

$$\vec{E}(\vec{r}) = \int_{S_I}\left[i\omega\mu_0(\hat{n}\times\vec{H})G + (\hat{n}\times\vec{E})\times\nabla' G + (\hat{n}\cdot\vec{E})\nabla' G\right]da' \quad (7)$$

so that the field can be seen as produced by a distribution of electric and magnetic current and electric charge densities. As in the case of scalar diffraction theory, the inconsistencies in Kirchhoff's result can be removed by a correct choice of Green's function, resulting in Smythe's diffraction theory:

$$\vec{E} = -\frac{1}{2\pi}\int_{S_I}\left(\vec{n}\times\vec{E}'\right)\times\vec{\nabla}\left(\frac{\exp(ikR)}{R}\right)da' \quad (8)$$

[1] Harmonic time dependence $e^{-i\omega t}$ is assumed throughout the paper.

Now the field can be seen as produced only by a magnetic current distribution. Eq. *(8)* implies that only the tangential components of the field in the aperture contribute to the diffracted field.

In general, most diffraction problems can be treated to a good extent using Kirchhoff's approximation of taking the field in the aperture to be the same as the incident field; however, this approximation fails when the size of the aperture or obstacle is small compared to the wavelength due to the effects of the edges.

This is also the case for a wave incident on an infinitely thin conducting screen with a subwavelength-scale circular aperture. The rest of the paper presents and discusses the solutions given by several authors.

II. BETHE'S SOLUTION

The solution proposed by Bethe [4] to the problem of diffraction from a small hole starts with the following observation: among the terms that appear in Kirchhoff's vectorial integral [eq. *(7)*] only the second term satisfies the boundary conditions. It is exactly this term that survives in Smythe's result [eq. *(8)*] which removes the inconsistencies in Kirchhoff's result. This same term can be seen as produced by a distribution of magnetic currents in the plane of the hole.

This observation makes it possible to adapt a method, which is similar to a perturbative calculation and which will be described below, for the solution of the problem.

The method considers the solution of the ("perturbed") problem as a superposition of the known field when there is no hole ("unperturbed" problem) with an unknown correction (scattered fields). If the plane of aperture is at *z=0*, then:

$$\begin{aligned} \vec{H} &= \vec{H}_0 + \vec{H}_1 \qquad z \le 0 \\ \vec{H} &= \vec{H}_2 \qquad z > 0 \end{aligned} \tag{9}$$

Here $\vec{H}_0$ is the magnetic field of the unperturbed solution, which is zero behind the screen, and $\vec{H}_1$ and $\vec{H}_2$ represent the correction terms. The electric field is decomposed in the same way.

The planar geometry of the problem imposes a particular symmetry on the correction fields:

$$\begin{aligned} E_{1x,y}(x,y,-z) &= E_{2x,y}(x,y,z) \\ H_{1x,y}(x,y,-z) &= -H_{2x,y}(x,y,z) \\ E_{1z}(x,y,-z) &= -E_{2z}(x,y,z) \\ H_{1z}(x,y,-z) &= H_{2z}(x,y,z) \end{aligned} \tag{10}$$

which makes it possible to obtain exact expressions for the boundary conditions of the correction fields in terms of the unperturbed fields:

$$\begin{aligned} E_{2n} &= \frac{1}{2} E_{0n} \\ H_{2\tan} &= \frac{1}{2} H_{0\tan} \end{aligned} \tag{11}$$

According to eq. *(8)* the diffracted field is determined by a distribution of magnetic currents in the plane of the hole. Simultaneously, the boundary conditions [eq. *(11)*] have to be fulfilled. The magnetic currents satisfy the continuity equation:

$$\nabla \cdot \vec{K} = i\omega\eta \tag{12}$$

where η is the magnetic charge density.

The electric and magnetic fields can be expressed as a function of potentials,

$$\begin{aligned} \vec{H} &= \varepsilon_0 \frac{\partial \vec{F}}{\partial t} - \vec{\nabla}\psi \\ \vec{E} &= \vec{\nabla} \times \vec{F} \end{aligned} \tag{13}$$

which are analogous to the vector and the scalar potentials. The potentials and fields, finally, can be written as a function of the magnetic charges and currents according to:

$$\begin{aligned} \psi &= \frac{1}{\mu_0} \int \eta(\vec{r}\,')\phi(|\vec{r} - \vec{r}\,'|)\, da' \\ \vec{F} &= -\int \vec{K}(\vec{r}\,')\phi(|\vec{r} - \vec{r}\,'|)\, da' \end{aligned} \tag{14}$$

where ϕ denotes the scalar Green's function.

The fields are calculated as:

$$\begin{aligned} \vec{H} &= \int \left[i\omega\varepsilon_0 \vec{K}(\vec{r}\,')\phi - \frac{1}{\mu_0}\eta(\vec{r}\,')\vec{\nabla}\phi \right] da' \\ \vec{E} &= \int \vec{K}(\vec{r}\,') \times \vec{\nabla}\phi \; da' \end{aligned} \tag{15}$$

These are the equations that link the magnetic currents and charges to the boundary conditions.

Up to now no approximations have been made. In order to completely solve the problem several approximations, based on the fact that the hole is small, have to be introduced.

The first approximation makes use of the fact that the hole is much smaller than the wavelength. Consequently, retardation can be neglected in the aperture. The problem then becomes analogous to an electrostatic problem since the harmonic time dependence is factored out. The second approximation assumes that the incident fields are constant in the hole if the hole is much smaller than the wavelength.

The challenge is to find the magnetic charges and currents that satisfy the boundary conditions for the tangential component of $\vec{H}$ [eq. *(11)*].

With the approximations outlined so far, the problem becomes equal to the well-known problem in electrostatics of finding a two-dimensional charge distribution that gives a constant field in the region occupied by the charges. It is known that a uniform distribution of dipoles inside an ellipsoid produces a constant field having the same direction as the dipoles. As shown in *Fig. 1*, the magnetic charge distribution in the hole is obtained by placing an ellipsoid with uniformly distributed magnetic dipoles oriented along the plane of the aperture in the hole and letting the semi-axis *b* go to zero.

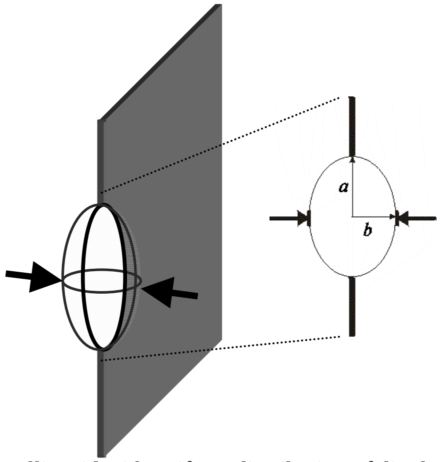

Fig. 1. Ellipsoid with uniform distribution of dipoles oriented along the plane of the aperture. The charge distribution in the hole is obtained by making the length of axis b very small.

The resulting magnetic charges and currents in the hole as determined by Bethe are:

$$\eta = -\frac{4\mu_0}{\pi\left(a^2 - r^2\right)^{1/2}}\vec{H}_0 \cdot \vec{r}$$
$$\vec{K}_m = \frac{i4\mu_0\omega}{\pi}\left(a^2 - r^2\right)^{1/2}\vec{H}_0 \qquad (16)$$

where a is the radius of the hole and r is the radial coordinate.

A similar calculation makes it possible to satisfy the boundary conditions for $\vec{E}$ [eq. *(11)*] without ruining the agreement already obtained for $\vec{H}$. Bethe obtains that an additional current is needed:

$$\vec{K}_E = \frac{2}{\pi\left(a^2 - r^2\right)^{1/2}}\vec{r}\,' \times \vec{E}_{0n} \qquad (17)$$

with $\vec{\nabla} \cdot \vec{K}_E = 0$ in order not to introduce any new charges in the boundary condition for $\vec{H}$.

There is a subtle mistake in this analysis that has been pointed out and corrected by Bouwkamp [5]: the result *(17*) by Bethe is derived on the assumption that the contribution of the current $\vec{K}_m$ [eq. *(16)*] to the electric field [eq. *(15)*] can be neglected to first order in *ka*. This assumption leads to inconsistencies in the boundary conditions.

Notwithstanding this mistake, the calculation by Bethe gives the correct far field as:

$$\vec{E}(\vec{r}) = \frac{1}{3\pi}k^2 a^3 \varphi_0 \hat{r} \times \left(2\vec{H}_0 \mu_0 c + \vec{E}_0 \times \hat{r}\right)$$
$$\vec{H}(\vec{r}) = \frac{1}{\mu_0 c}\hat{r} \times \vec{E} \qquad (18)$$

These fields are of order $k^2 a^3$. The total radiated power is given by:

$$P_{tot} = \frac{4c}{27\pi}k^4 a^6\left(4\mu_0 H_0{}^2 + \varepsilon_0 E_0{}^2\right) \qquad (19)$$

This result has to be contrasted with Kirchhoff's solution, which gives:

$$P_{tot} \sim k^2 a^4 \qquad (20)$$

The conclusion is that the power radiated by a small aperture is much less than what Kirchhoff's theory predicts.

An interesting feature of the radiation from a small aperture is that the far field is equivalent to that of a linear combination of a magnetic dipole and an electric dipole, which expressed in terms of the incident field are given by:

$$\vec{M} = -\frac{8}{3}a^3 \vec{H}_0$$
$$\vec{P} = -\frac{4}{3}\varepsilon_0 a^3 \vec{E}_0 \qquad (21)$$

For normal incidence only the magnetic dipole is necessary, since $\vec{E}_0$ becomes zero.

III. ANALYSIS OF BETHE'S SOLUTION

Careful examination of Bethe's solution reveals that the electric field is discontinuous in the hole, contrary to what is required by the boundary conditions. In the case of a unit amplitude electric field with x-polarization incident on the aperture the boundary conditions are given by:

$$E_x = 0,\ E_y = 0,\ H_z = 0 \quad \text{on the screen}$$
$$H_x = 0,\ H_y = (\mu_0 c)^{-1},\ E_z = 0 \quad \text{in the hole.} \qquad (22)$$

As Bouwkamp [5] shows, Bethe's first order approximation does not satisfy the condition $E_z=0$ in the hole.

The mistake made by Bethe in his analysis is that the plane that contains the hole is not closed due to the boundary introduced by the presence of the aperture, so that it is necessary to include the contributions of line sources along the edge of the hole. It has been shown by Bouwkamp that such contribution can be ignored if an additional boundary condition is placed on the electric field: its component tangential to the edge (E_s) must vanish as the square root of the distance from it, such that:

$$E_s = 0 \qquad (23)$$

at the edge of the hole. An additional condition for the electric field is that the component normal to the edge becomes infinite as the inverse square root of the distance from the edge.

For normal incidence, Bethe's first order approximation for the magnetic surface-charge and surface-current densities [eqs. *(16)*] become:

$$\eta = -\frac{8}{c\pi}\frac{y}{\sqrt{a^2 - x^2 - y^2}} \qquad (24)$$

$$K_x = 0,\ K_y = \frac{i8k}{\pi}\sqrt{a^2 - x^2 - y^2} \qquad (25)$$

The expression for the magnetic charge density is correct; however, the one for the magnetic surface-current density is not, since the solution to eq. *(12)* is not unique. In order to obtain a unique solution, it is necessary to impose the additional boundary condition, eq. *(23)*.

Using eq. *(24)* in eq. *(14)*, the scalar potential is given by:

$$\psi = -\frac{2}{\pi^2}\frac{1}{\mu_0 c}\int \frac{y'}{\sqrt{a^2 - x'^2 - y'^2}}\frac{dx'dy'}{\sqrt{(x - x')^2 + (y - y')^2 + z^2}} \qquad (26)$$

Such integral can be solved by noting that it represents the solution to a three-dimensional potential equation and by making an expansion in terms of oblate-spheroidal potential functions [6] (See Appendix I). Substituting eq. *(25*) in eq. *(14)* a similar integral is obtained for the vector potential $\vec{F}$:

$$F_y = -\frac{2ik}{\pi^2}\iint\sqrt{a^2 - x'^2 - y'^2}\,\frac{dx'dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}} \qquad (27)$$

Solving the integrals given by eqs. *(26)* and *(27)* the fields in the hole can be obtained:

$$\begin{aligned} E_x &= -\frac{4}{\pi}ik\sqrt{a^2 - x^2 - y^2} & H_x &= 0 \\ E_y &= 0 & H_y &= \frac{1}{\mu_0 c} \\ E_z &= ikx & H_z &= -\frac{4}{\mu_o c\pi}\frac{y}{\sqrt{a^2 - x^2 - y^2}} \end{aligned} \qquad (28)$$

It can be seen that E_z is not equal to zero as required by the boundary conditions [eq. *(22)*], so that Bethe's solution is not a correct solution to the diffraction problem. In his analysis Bethe assumes that the contribution from K_m [eq. *(16)*], which is of order *ka*, as eq. *(28)* shows, is negligible. Another thing to notice is that the component of the electric field tangential to the edge does not become infinite as expected. Eq. *(28)* shows that the electric field is an order of magnitude *ka* smaller than the magnetic field.

IV. BOUWKAMP'S SOLUTION

Bouwkamp's approach [7] in solving the diffraction problem consists in first deriving the solution for a disk and then using Babinet's principle to obtain the magnetic currents for the case of the aperture. [2]

Electric currents are induced on the disk due to the interaction with the incident fields. Once such currents are known, the vector potential can be obtained through:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \vec{I}\,\frac{\exp(ikR)}{R}\,da\,, \qquad (29)$$

which in the Lorentz gauge gives the scattered fields:

$$\vec{H}^s = \frac{1}{\mu_0}\nabla\times\vec{A} \qquad \vec{E}^s = i\omega\vec{A} - \frac{c}{ik}\nabla\left(\nabla\cdot\vec{A}\right). \qquad (30)$$

Using the appropriate boundary conditions for the problem:

$$E_x^s = -1,\ E_y^s = 0,\ H_z^s = 0 \quad (x^2 + y^2 \le a) \qquad (31)$$

combined with eqs. *(29)* and *(30)* a system of integro-differential equations is obtained for the fields in the aperture:

$$\begin{aligned} &\frac{\partial A_x}{\partial y} = \frac{\partial A_y}{\partial y} \\ &\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + k^2 A_x = ik \\ &\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + k^2 A_y = 0 \end{aligned} \quad \text{and} \quad \begin{aligned} A_x &= \frac{\mu_0}{4\pi}\int I_x\,\frac{\exp(ikR)}{R}\,da \\ A_y &= \frac{\mu_0}{4\pi}\int I_y\,\frac{\exp(ikR)}{R}\,da \end{aligned} \qquad (32)$$

This system of equations cannot be solved analytically. In order to obtain a solution, it is necessary to use the symmetry of the problem and impose the additional boundary conditions at the edge of the disk.

Since a plane wave at normal incidence is being considered, the problem has axial symmetry. This together with the boundary conditions at the edge of disk implies that the behavior of the electric current density must satisfy:

$$\frac{1}{c}I_x = \frac{A(\rho) + B(\rho)\cos 2\varphi}{\sqrt{1-\rho^2}},\ \frac{1}{c}I_y = \frac{B(\rho)}{\pi^2\sqrt{1-\rho^2}}\sin 2\varphi \qquad (33)$$

from which the following condition can be obtained:

$$I_y = -\frac{1}{2}\frac{\partial I_x}{\partial\varphi} = \frac{1}{2}\left(y\frac{\partial I_x}{\partial x} - x\frac{\partial I_x}{\partial y}\right) \qquad (34)$$

A similar condition can be obtained for the vector potential. These relations allow eqs. *(32)* to be reduced to an integral equation containing only one variable (I_x or I_y), so that through a series expansion the coefficients *A* and *B* of eqs. *(33)* can be obtained.

Through the use of Babinet's principle the electric currents will act as magnetic current densities, which to first order are given by:

$$K_y = \frac{8ik}{3\pi}\frac{2a^2 - x^2 - 2y^2}{\sqrt{a^2 - x^2 - y^2}},\ K_x = -\frac{8ik}{3\pi}\frac{xy}{\sqrt{a^2 - x^2 - y^2}} \qquad (35)$$

The fields in the aperture can now be obtained:

$$\begin{aligned} E_x &= -\frac{4ik}{3\pi}\frac{2a^2 - x^2 - 2y^2}{\sqrt{a^2 - x^2 - y^2}} & H_x &= 0 \\ E_y &= -\frac{4ik}{3\pi}\frac{xy}{\sqrt{a^2 - x^2 - y^2}} & H_y &= \frac{1}{\mu_0 c} \\ E_z &= 0 & H_z &= -\frac{4}{\mu_0 c\pi}\frac{y}{\sqrt{a^2 - x^2 - y^2}} \end{aligned} \qquad (36)$$

These equations can now be directly compared to Bethe's results [eqs. *(28)*]. As can be seen, the expressions for the magnetic fields are the same, which is due to the fact that to first order there is no contribution from the magnetic current density to the magnetic fields.

A contour plot of both Bethe's and Bouwkamp's results can be seen in *Fig.2* and *Fig. 3*, respectively. In these plots, the lighter regions represent higher intensity.

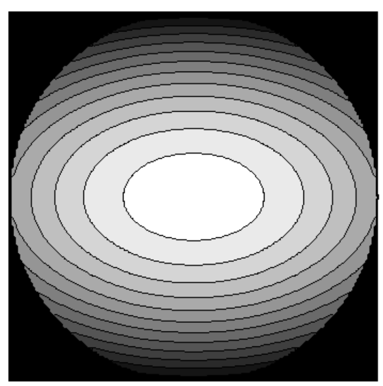

Fig. 2. Total electric field $|\vec{E}|^2$ *in the aperture obtained by Bethe.*

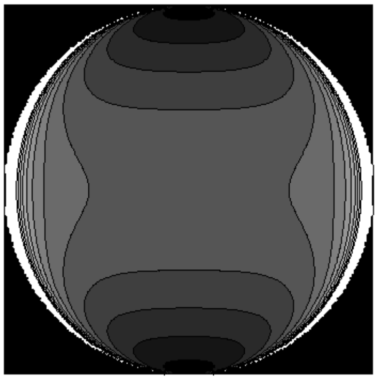

Fig. 3. Total electric field $\left|\vec{E}\right|^2$ in the aperture obtained by Bouwkamp.

There are a few things worth noting about these results. The first is that Bethe's result remains finite throughout the aperture, contrary to what is expected. The other is that Bouwkamp's solution becomes infinite on the edge of the aperture, except at $x=a$ and $x=-a$. This result is expected, since the incident field is x polarized and thus has no normal components to the edge at these points.

An important result found by Bouwkamp in his analysis is that Bethe's expressions for the far field are correct.

V. FLAMMER'S SOLTUION

Flammer [8, 9] follows the method of separation of variables to obtain an exact solution for the complementary problem of diffraction from a perfectly conducting disk. Diffraction from a hole is then derived through the use of the Babinet's principle, as was done by Bouwkamp.

The method of separation of variables is applicable to the solution of the scalar Helmholtz equation with some prescribed boundary conditions. A system of coordinates in which the boundary is described by a constant coordinate surface is needed in order to solve the problem. For the problem of the disk such a system is the oblate spheroidal coordinates system. (See Appendix I)

There is, however, a problem that needs to be addressed. The electromagnetic field is a vectorial solution of Helmholtz equation that has zero divergence; multiplying a scalar solution by a constant vector gives an acceptable electromagnetic field in the case of the plane waves but not in the case of spheroidal waves.

A mathematical prescription that solves this problem is the following: take a solution of the Helmholtz equation, ψ, and calculate the vector field $\vec{A}=\vec{\nabla}\times(\vec{a}\psi)$. If $\vec{a}=K_1\vec{e}$ or $\vec{a}=K_1\vec{r}$, where K_1 is a constant, $\vec{e}$ is a constant vector, and $\vec{r}$ is the position vector, then $\vec{A}$ has the required properties. The ψ's in the present case are the complete collection of spheroidal waves.

Expanding Flammer's solution in powers of ka the terms found by Bouwkamp are recovered.

APPENDIX I. OBLATE-SPHEROIDAL COORDINATES

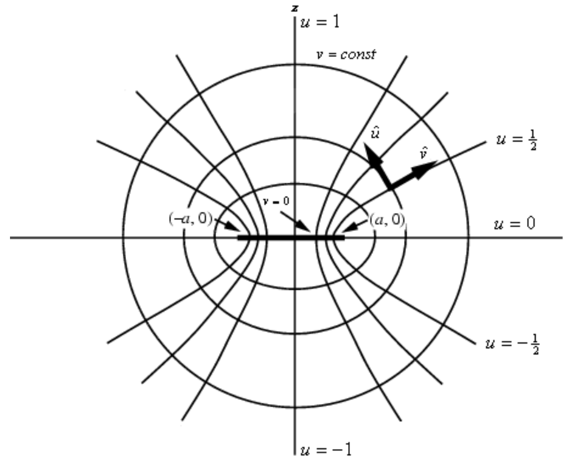

Fig. 4. Oblate-Spheroidal coordinate system.

The oblate-spheroidal coordinates system [10] is defined by:

$$\begin{aligned} z &= auv \\ x &= a\sqrt{(1-u^2)(1+v^2)}\cos\varphi \\ y &= a\sqrt{(1-u^2)(1+v^2)}\sin\varphi \end{aligned} \tag{37}$$

where $-1\le u\le 1,\ 0\le v\le\infty,\ 0\le\varphi<2\pi$.

The definition of u, v, and φ can be seen in *Fig. 4*. The system is obtained by rotating *Fig. 4* around the z-axis, so that the surface $v=const.$ is an oblate ellipsoid of revolution, the surface $u=const.$ corresponds to a hyperboloid, and φ is just the angle with respect to the x-axis in the x-y plane.

Using this coordinate system, the disk is represented by the coordinate $v=0$, while in the complementary problem of the circular aperture the screen is given by $u=0$.

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