Welcome again!

Nano-optics studies light-matter interactions on the sub-wavelength scale. The goal of this course is to quantitatively understand the fundamental concepts of nano-optics, including:

- Superresolution microscopy
- Quantum light sources
- Optical antennas
- Optical forces
- ...
Administrative details: Lab/Research Projects

• First name/ last name?

• Check website for group assignments
• Find your team-mates
• Determine a spokesperson to contact your supervisor
Why nano-optics?

- The energy scale of our interest corresponds to about 1 µm wavelength.
- The length scales of scientific and technological interest are approaching the atomic scale.
  Read Feynman’s talk “There is plenty of room at the bottom.”
  ➔ We need to control electromagnetic fields and their interaction with matter at sub-λ scales.

Thermal noise

kT

100 meV

1 nm

10 eV

1 m

LIFE

Size mismatch

Ry
On the menu today

• Motivation: Why nano-optics?
• Repetition: electromagnetism
• Optical imaging:
  • Focusing by a lens
    • Angular spectrum
    • Paraxial approximation
    • Gaussian beams
    • Method of stationary phase
• The diffraction limit
• Fluorophores
• Example: Fluorescence microscopy
• Example: STED microscopy
• Example: Localization microscopy
• Example: Scanning probe microscopy
How does focusing by a lens work?

What does the field distribution here look like?
How does focusing by a lens work?
How does focusing by a lens work?

$$E(x, z=0) = E_0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{ikx \sin[\theta_1]}$$

$$\theta_1 = \pm 80^\circ$$
How does focusing by a lens work?

\[ E(x, z=0) = E_0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{ikx \sin[\theta_1]} \]

\[ \theta_1 = 0^\circ, \pm 15^\circ, \pm 30^\circ, \pm 45^\circ, \pm 60^\circ, \pm 75^\circ \]

+ apodization
Angular spectrum

Physics:

\[(\nabla^2 + k^2) \mathbf{E}(\mathbf{r}) = 0 \quad \rightarrow \quad \hat{\mathbf{E}}(k_x, k_y; z) = \hat{\mathbf{E}}(k_x, k_y; 0) e^{\pm i k_z z}\]

MATH:

\[
\hat{\mathbf{E}}(k_x, k_y; z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{E}(x, y, z) e^{-i (k_x x + k_y y)} \, dx \, dy
\]

\[
\mathbf{E}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x, k_y; z) e^{i (k_x x + k_y y)} d k_x \, d k_y
\]

Together:

\[
\mathbf{E}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x, k_y; 0) e^{i (k_x x + k_y y \pm k_z z)} \, d k_x \, d k_y
\]
Angular spectrum

\[ \nabla^2 + k^2 \mathbf{E}(\mathbf{r}) = 0 \quad \longrightarrow \quad \hat{\mathbf{E}}(k_x, k_y; z) = \hat{\mathbf{E}}(k_x, k_y; 0) e^{\pm ik_z z} \]

Together:

\[ \mathbf{E}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x, k_y; 0) e^{i[k_xx + k_y y \pm k_z z]} \, dk_x \, dk_y \]
Paraxial approximation

\[ \mathbf{E}(x, y, z) = \int_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x, k_y; 0) e^{i[k_x x + k_y y \pm k_z z]} \, dk_x \, dk_y \]

mit

\[ k_z = k \sqrt{1 - (k_x^2 + k_y^2)/k^2} \approx k - \frac{(k_x^2 + k_y^2)}{2 k} \]

\[ k_z = k \cos \theta = k [1 - \theta^2/2 + ..] \]

\[ \rightarrow \quad \text{Fields propagate predominantly in z-direction!} \]
Gaussian beams

\[ E(x', y', 0) = E_0 e^{-\frac{x'^2 + y'^2}{w_0^2}} \]

\[ \rightarrow \quad \hat{E}(k_x, k_y; 0) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} E_0 e^{-\frac{x'^2 + y'^2}{w_0^2}} e^{-i[k_xx' + k_y y']} \, dx' \, dy' \]

\[ = E_0 \frac{w_0^2}{4\pi} e^{-(k_x^2 + k_y^2)\frac{w_0^2}{4}} \]

\[ \rightarrow \quad E(x, y, z) = E_0 \frac{w_0^2}{4\pi} e^{ikz} \iint_{-\infty}^{\infty} e^{-(k_x^2 + k_y^2)\left(\frac{w_0^2}{4} + \frac{i\,z}{2\,k}\right)} e^{i[k_xx + k_y y]} \, dk_x \, dk_y \]

\[
\int_{-\infty}^{\infty} \exp(-a x^2 + i b x) \, dx = \sqrt{\pi/a} \exp(-b^2/4a) \\
\int_{-\infty}^{\infty} x \exp(-a x^2 + i b x) \, dx = i b \sqrt{\pi} \exp(-b^2/4a) / (2a^{3/2})
\]
Gaussian Beams

\[ E(\rho, z) = E_0 \frac{w_o}{w(z)} e^{-\frac{\rho^2}{w^2(z)}} e^{i[kz-\eta(z)+k\rho^2/2R(z)]} \]

- \( w(z) = w_o \left(1 + \frac{z^2}{z_o^2}\right)^{1/2} \) \quad Waist Radius
- \( R(z) = z \left(1 + \frac{z^2}{z_o^2}\right) \) \quad Wavefront Radius
- \( \eta(z) = \arctan \frac{z}{z_o} \) \quad Phase Correction

\[ z_o = \frac{k w_o^2}{2} \] \quad Rayleigh Range

Note that we started with ONLY the field distribution in the plane \( z=0 \)!
Gaussian Beams

\[ z_o = \frac{k w_o^2}{2} \quad \theta = \frac{2}{k w_o} \]
A better description of focused fields

Can we measure this field distribution?
Mapping the field distribution in the focus

fluorescence rate $\sim$ excitation rate

contrast $\sim |\mu \cdot E(x,y;z_0)|^2$

Detector has no spatial resolution
Fluorescent molecules – Jablonski diagram
Mapping the field distribution in the focus

Map of focal intensity distribution

Detector has no spatial resolution

\[ \text{fluorescence rate} \sim \text{excitation rate} \]

\[ \text{contrast} \sim |\mu \cdot E(x,y;z_0)|^2 \]
Mapping the field distribution in the focus

Can we calculate this field distribution?

fluorescence rate $\sim$ excitation rate

contrast $\sim |\mu \cdot E(x,y;z_0)|^2$
Far-field

\[ E_{\infty}(s_x, s_y) = -2\pi i k s_z \hat{E}(k s_x, k s_y; 0) \frac{e^{ikr}}{r} \]

\[
\left( \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) = \left( \frac{k_x}{k}, \frac{k_y}{k}, \frac{k_z}{k} \right)
\]
\[ \hat{E}(k_x, k_y; 0) = \frac{E_0}{4\pi^2} \int_{-L_y}^{+L_y} \int_{-L_x}^{+L_x} e^{-i[k_x x' + k_y y']} \, dx' \, dy' \]

\[ = E_0 \frac{L_x L_y}{\pi^2} \frac{\sin(k_x L_x)}{k_x L_x} \frac{\sin(k_y L_y)}{k_y L_y} \]

\[ E_\infty(s_x, s_y) = -iks_z E_0 \frac{2L_x L_y}{\pi} \frac{\sin(k s_x L_x)}{k s_x L_x} \frac{\sin(k s_y L_y)}{k s_y L_y} \frac{e^{ikr}}{r} \]
Angular spectrum in terms of far-field

\[ E(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{E}(k_x, k_y; 0) e^{i[k_x x + k_y y \pm k_z z]} \, dk_x \, dk_y \]

\[ = \frac{i r e^{-i kr}}{2\pi k_z} E_{\infty}(k_x, k_y) \]

From method of stationary phase:

\[ E(x, y, z) = \frac{i r e^{-i kr}}{2\pi} \int \int E_{\infty}(\frac{k_x}{k}, \frac{k_y}{k}) e^{i[k_x x + k_y y \pm k_z z]} \frac{1}{k_z} \, dk_x \, dk_y \]

\[ k_x \rightarrow k s_x \]

\[ k_y \rightarrow k s_y \]

For \( k_z \sim k \): Fourier Optics!
Back to the lens

- We can calculate the field near a focus if we just know the far-field

\[ E(x, y, z) = \frac{ir e^{-ikr}}{2\pi} \int \int E_\infty \left( \frac{k_x}{k}, \frac{k_y}{k} \right) e^{i[k_x x + k_y y \pm k_z z]} \frac{1}{k_z} \, dk_x \, dk_y \]
So what does a lens do?

\[ E(x, y, z) = \frac{ir e^{-ikr}}{2\pi} \int \int \left[ \frac{1}{k_z} \right] E_\infty \left( \frac{k_x}{k}, \frac{k_y}{k} \right) e^{i[k_x x + k_y y + k_z z]} \cdot \frac{1}{k_z} \, dk_x \, dk_y \]
So what does a lens do?

Sine Condition (aplanatic system)

\[ h = f \sin(\theta) \]

Ray Continuity (energy conservation)

\[ |E_2| = |E_1| \sqrt{\frac{n_1}{n_2}} \sqrt{\frac{\mu_2}{\mu_1}} (\cos \theta)^{1/2} \]
So what does a lens do?

\[ h = f \sin(\theta) \]

What about this term?

\[ E(x, y, z) = \frac{ir e^{-ikr}}{2\pi} \int \int E_\infty \left( \frac{k_x}{k}, \frac{k_y}{k} \right) e^{i[k_x x + k_y y \pm k_z z]} \frac{1}{k_z} \, dk_x \, dk_y \]

\[ |E_2| = |E_1| \sqrt{n_1} \sqrt{\frac{n_2}{\mu_1}} \sqrt{\frac{\mu_2}{\mu_1}} \cos^1/2 \theta \]
Project plane on sphere

\[ dk_x \, dk_y = \cos \theta \left[ k^2 \sin \theta \, d\theta \, d\phi \right] \]

\[ \frac{1}{k_z} \, dk_x \, dk_y = k \, \sin \theta \, d\theta \, d\phi \]
So what does a lens do?

\[ h = f \sin(\theta) \]

\[ |E_2| = |E_1| \sqrt{\frac{n_1}{n_2}} \sqrt{\frac{\mu_2}{\mu_1}} (\cos \theta)^{1/2} \]
Field after lens

\[ E_\infty = \left[ t^s \left[ E_{inc} \cdot n_\phi \right] n_\phi + t^p \left[ E_{inc} \cdot n_\rho \right] n_\theta \right] \sqrt{\frac{n_1}{n_2}} (\cos \theta)^{1/2} \]
Angular spectrum representation

\[ E(x, y, z) = \frac{ir e^{-ikr}}{2\pi} \int \int E_\infty \left( \frac{k_x}{k}, \frac{k_y}{k} \right) \ e^{i[k_x x + k_y y \pm k_z z]} \ \frac{1}{k_z} \ dk_x \ dk_y \]

Change coordinates

Coordinates in focal region

\[ E(\rho, \varphi, z) = \frac{ikf e^{-ikf}}{2\pi} \int_{0}^{\theta_{max}} \int_{0}^{2\pi} E_\infty(\theta, \phi) \ e^{ikz \cos \theta} e^{ik\rho \sin \theta \cos(\phi - \varphi)} \sin \theta \ d\phi \ d\theta \]

Coordinates on reference sphere

NA
Simplest case: Focusing of $(0,0)$-Gaussian beam

$$E_{\text{inc}} = E_{\text{inc}} n_x \quad t_\theta = t_\phi = 1$$

$$E_\infty(\theta, \phi) = E_{\text{inc}}(\theta, \phi) \left[ \cos \phi n_\theta - \sin \phi n_\phi \right] \sqrt{n_1/n_2} (\cos \theta)^{1/2}$$

$$= E_{\text{inc}}(\theta, \phi) \frac{1}{2} \left[ \begin{array}{c} (1 + \cos \theta) - (1 - \cos \theta) \cos 2\phi \\ - (1 - \cos \theta) \sin 2\phi \\ -2 \cos \phi \sin \theta \end{array} \right] \sqrt{n_1/n_2} (\cos \theta)^{1/2}$$

$(0, 0)$ mode:

$$E_{\text{inc}} = E_o e^{-(x_\infty^2 + y_\infty^2)/w_o^2} = E_o e^{-f^2 \sin^2 \theta/w_o^2}$$

Let’s skip some lengthy coordinate transformations and integrations...
Integrate over $\phi$

\[
\begin{align*}
\int_0^{2\pi} \cos n\phi \, e^{ix \cos(\phi - \varphi)} \, d\phi &= 2\pi (i^n) J_n(x) \cos n\varphi \\
\int_0^{2\pi} \sin n\phi \, e^{ix \cos(\phi - \varphi)} \, d\phi &= 2\pi (i^n) J_n(x) \sin n\varphi .
\end{align*}
\]
The solution... is a bit lengthy

\[ E(\rho, \varphi, z) = \frac{ikf}{2} \sqrt{\frac{n_1}{n_2}} E_o e^{-ikf} \begin{bmatrix} I_{00} + I_{02} \cos 2\varphi \\ I_{02} \sin 2\varphi \\ -2iI_{01} \cos \varphi \end{bmatrix} \]

\[ H(\rho, \varphi, z) = \frac{ikf}{2Z_{\mu\varepsilon}} \sqrt{\frac{n_1}{n_2}} E_o e^{-ikf} \begin{bmatrix} I_{02} \sin 2\varphi \\ I_{00} - I_{02} \cos 2\varphi \\ -2iI_{01} \sin \varphi \end{bmatrix} \]

Apodization function:

\[ f_w(\theta) = e^{-\frac{1}{f_0^2} \frac{\sin^2 \theta}{\sin^2 \theta_{max}}} \]

\[ I_{00} = \int_{0}^{\theta_{max}} f_w(\theta) (\cos \theta)^{1/2} \sin \theta (1 + \cos \theta) J_0(k\rho \sin \theta) e^{ikz \cos \theta} d\theta \]

\[ I_{01} = \int_{0}^{\theta_{max}} f_w(\theta) (\cos \theta)^{1/2} \sin^2 \theta J_1(k\rho \sin \theta) e^{ikz \cos \theta} d\theta \]

\[ I_{02} = \int_{0}^{\theta_{max}} f_w(\theta) (\cos \theta)^{1/2} \sin \theta (1 - \cos \theta) J_2(k\rho \sin \theta) e^{ikz \cos \theta} d\theta \]
Contour plots of constant $|E|^2$ in the focal region of a focused Gaussian beam (NA = 1.4, $n = 1.518$, $f_0 = 1$): (a) in the plane of incident polarization $(x, z)$; (b) in the plane perpendicular to the plane of incident polarization $(y, z)$. A logarithmic scaling is used, with a factor of 2 difference between adjacent contour lines. Images (c), (d), and (e) show the magnitudes of the individual field components $|E_x|^2$, $|E_y|^2$, and $|E_z|^2$, respectively, in the focal plane ($z = 0$).
Strongly focused Gaussian beam

Influence of the filling factor $f_0$ of the back-aperture on the sharpness of the focus. A lens with NA = 1.4 is assumed and the index of refraction is 1.518. The figure shows the magnitude of the electric field intensity $|E|^2$ in the focal plane $z = 0$. The dashed curves have been evaluated along the $x$-direction (plane of polarization) and the solid curves along the $y$-direction. All curves have been scaled to an equal amplitude. The scaling factor is indicated in the figures. The larger the filling factor, the bigger the deviation between the solid and dashed curves, indicating the importance of polarization effects.
Weakly focused beam

- Assume strongly overfilled back-aperture
- Assume small NA

\[ E_{\text{inc}} = E_{\text{inc}} \, n_x \quad t_\theta^s = t_\theta^p = 1 \]

Focal plane (z=0):

\[ I_{00} \approx \frac{2}{k \rho} \int_0^{k \rho \theta_{\text{max}}} x J_0(x) \, dx = 2 \theta_{\text{max}}^2 \frac{J_1(k \rho \theta_{\text{max}})}{k \rho \theta_{\text{max}}} \]

\[ E \approx ikf \, \theta_{\text{max}}^2 \, E_0 \, e^{-ikf} \, \frac{J_1(k \rho \theta_{\text{max}})}{k \rho \theta_{\text{max}}} \, n_x \]

Not Gaussian!

Why is this a jinc?
Mapping the field distribution in the focus

Map of focal intensity distribution

Detector has no spatial resolution

fluorescence rate \sim \text{excitation rate}

\text{contrast} \sim |\mu \cdot \mathbf{E}(x,y;z_o)|^2
Single-molecule detection

Single-molecule excitation strongly focused laser beam excitation rate in each pixel absorption dipole moment

...reconstructed from the recorded patterns. Compare the patterns marked $x$, $y$, and $z$ with those in Fig. 3.11.

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What does the image of a point-source look like?
Point-spread function
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• Example: Scanning probe microscopy
Point-spread function

\[ k_x = k_2 \sin \theta_2 \cos \phi \]
\[ k_y = k_2 \sin \theta_2 \sin \phi \]
\[ k_z = k_2 \cos \theta_2 \]

\[ f_1 \sin \theta_1 = f_2 \sin \theta_2 \]

\[ x = \rho \cos \varphi \]
\[ y = \rho \sin \varphi \]
Angular spectrum

\[
\mathbf{E}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\mathbf{E}}(k_x, k_y; 0) \ e^{i [k_x x + k_y y \pm k_z z]} \ dk_x \ dk_y
\]

\[
\mathbf{E}(x, y, z) = \frac{ir e^{-ikr}}{2\pi} \int_{(k_x^2 + k_y^2) \leq k^2} \int \mathbf{E}_\infty(k_x, k_y) \ e^{i [k_x x + k_y y \pm k_z z]} \ \frac{1}{k_z} \ dk_x \ dk_y
\]

\[
\mathbf{E}(\rho, \varphi, z) = \frac{ik_2 f_2 e^{-ik_2 f_2}}{2\pi} \ \text{Max}[\theta_2] \ 2\pi \ \int_{0}^{\text{Max}[\theta_2]} \int_{0}^{2\pi} \mathbf{E}_\infty(\theta_2, \phi) \ e^{ik_2 \rho \cos(\phi - \varphi)} \ e^{ik_2 \rho \sin(\theta_2 \cos(\phi - \varphi))} \sin \theta_2 \ d\varphi \ d\theta_2
\]

Farfield of dipole:

\[- \frac{p k_1^2}{4\pi \varepsilon_0 \varepsilon_1} \ \frac{\exp(ik_1 f_1)}{f_1} \ \cos \theta\]
Paraxial approximation

\[ \sin \theta_1 \approx \theta_1 \]
\[ \sin \theta_2 \approx \theta_2 \]

\[ f_1 \sin \theta_1 = f_2 \sin \theta_2 \]

\[ E(\rho, \varphi, z) = \frac{i k_2 f_2 e^{-i k_2 f_2}}{2\pi} \int_0^{\text{Max}[\theta_2]} \int_0^{2\pi} E_\infty(\theta_2, \phi) e^{ik_2 z \cos \theta_2} e^{ik_2 \rho \sin \theta_2 \cos(\phi-\varphi)} \sin \theta_2 \, d\phi \, d\theta_2 \]

\[ \int_0^{2\pi} e^{ix \cos(\phi-\varphi)} \, d\phi = 2\pi J_0(x) \quad \longrightarrow \quad E(\rho, \varphi, z) \propto \int_0^{\text{Max}[\theta_2]} e^{ik_2 z \cos \theta_2} J_0(k_2 \rho \sin \theta_2) \sin \theta_2 \, d\theta_2 \]

\[ \int x J_0(x) \, dx = x J_1(x) \quad \longrightarrow \quad \lim_{\theta_{\text{max}} \ll \pi/2} \left| E(\rho, z=0) \right|^2 = \frac{\pi^4}{\varepsilon_0^2 n_1 n_2} \frac{p^2}{\lambda^6} \frac{\text{NA}^4}{M^2} \left[ 2 \frac{J_1(2\pi \tilde{\rho})}{(2\pi \tilde{\rho})} \right]^2 \]

\[ \tilde{\rho} = \frac{\text{NA} \rho}{M \lambda} \]

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Point-spread function

\[
\lim_{\theta_{\text{max}} \ll \pi/2} |E(\rho, z=0)|^2 = \frac{\pi^4}{\varepsilon_0^2 n_1 n_2} \frac{p^2}{\lambda^2} \frac{NA^4}{M^2} \left[ 2 \frac{J_1(2\pi \tilde{\rho})}{(2\pi \tilde{\rho})} \right]^2
\]

\[\tilde{\rho} = \frac{NA \rho}{M \lambda}\]
Point-spread function
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Classical resolution limit

\[ \Delta k_{\parallel} \Delta r_{\parallel} \geq 1 \quad \text{Min} [\Delta r_{\parallel}] = \frac{\lambda}{4\pi n} \quad \text{Min} [\Delta r_{\parallel}] = \frac{\lambda}{4\pi NA} \]

Abbe (1873) : \quad \text{Min} [\Delta r_{\parallel}] = 0.6098 \frac{\lambda}{NA}

E. Abbe, Arch. Mikrosk. Anat. 9, 413 (1873).
Abbe’s Resolution Limit

E. Abbe, Arch. Mikrosk. Anat. 9, 413 (1873).