Homework 3

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Nano Optics, Fall Semester 2018
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The goal of this problem set is to understand how the local density of optical states (LDOS) can be used to modify the properties of a light source. As a practical example, we consider how to maximize the light output of a light-emitting diode.

Please stick to the following points:

- Hand in your solution on time (beginning of lecture) in legible hand-written form or typeset using LaTeX (do not provide any code of any programming language, do not send your solutions by email).
- Keep your solutions in the order of the problem set.
- Show explicitly how you arrived at your answer and comment on your thoughts. Focus on the presentation of your solutions as if you were to explain them to a reader who is not familiar with the topic. Be brief but to the point.
- If you decide to generate graphs with a computer (which is encouraged), please position them at the appropriate position within your solution.
- Always label graphs and axes and provide units where appropriate.
- You can receive a maximum of 100 points for the correct solutions to the problems. For the quality of your presentation, you can receive an additional 10 points.

You are strongly encouraged to discuss the problems with your peers. In case of remaining questions, feel free to approach the course instructors.

1 Drexhage's experiment and the local density of optical states [80 pts.]

The power radiated by a dipole depends on the dipole's environment via the local density of optical states, also referred to as the radiation resistance. For a quantum light source, it is the decay rate that can be engineered via the LDOS. In this exercise, we work through an example illustrating the *local* character of the density of states.
Drexhage found characteristic oscillations in the decay rate of quantum emitters in front of a metallic mirror as a function of emitter-mirror distance, which is a signature of a spatial variation in the LDOS. Model his experiment by considering a dipole in front of a perfectly reflecting mirror. Let the mirror surface be in the xy-plane, locate the emitter at $\boldsymbol{r}_0$ at a distance d on the positive z-axis, and let the half-space along $+z$ be filled with a dielectric medium of refractive index n.

(a) (12 pts.) The free space Green function in a linear, isotropic and homogeneous medium reads

$$\overleftrightarrow{\boldsymbol{G}}_0(\boldsymbol{r}, \boldsymbol{r'}) = \frac{\exp[\mathrm{i}kR]}{4\pi R}\left[\left(1 + \frac{\mathrm{i}kR - 1}{k^2R^2}\right)\overleftrightarrow{\boldsymbol{I}} + \frac{3 - 3\mathrm{i}kR - k^2R^2}{k^2R^2}\frac{\boldsymbol{RR}}{R^2}\right], \tag{1}$$

with $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$. Calculate $\lim_{\boldsymbol{r'}\to\boldsymbol{r}} \mathrm{Im}\overleftrightarrow{\boldsymbol{G}}_0(\boldsymbol{r}, \boldsymbol{r'})$.
Hint: Argue carefully how you treat the limit of the factor $\frac{\boldsymbol{RR}}{R^2}$.

(b) (8 pts.) Set up the Green function $\overleftrightarrow{\boldsymbol{G}}(\boldsymbol{r}, \boldsymbol{r}_0) = \overleftrightarrow{\boldsymbol{G}}_0(\boldsymbol{r}, \boldsymbol{r}_0) + \overleftrightarrow{\boldsymbol{G}}_s(\boldsymbol{r}, \boldsymbol{r}_0)$ of Drexhage's system as a superposition of the vacuum Green function $\overleftrightarrow{\boldsymbol{G}}_0$ and the scattered part generated by the mirror $\overleftrightarrow{\boldsymbol{G}}_s$. In order to express $\overleftrightarrow{\boldsymbol{G}}_s$ through $\overleftrightarrow{\boldsymbol{G}}_0$, remember that the boundary conditions at a perfect mirror can be fulfilled by considering appropriate image charges. Make a sketch, both for a dipole oriented parallel and perpendicularly to the mirror to support your argument.

(c) (8 pts.) Calculate the decay rate enhancement $\gamma_{||}/\gamma_0$ as a function of mirror-emitter distance for an emitter oriented parallel to the mirror surface, where γ_0 is the emitter's decay rate in vacuum..

(d) (6 pts.) Plot your result for $\gamma_{||}/\gamma_0$ as a function of normalized emitter mirror distance $kd = 2\pi nd/\lambda$ up to $10kd$.

(e) (4 pts.) Give an intuitive explanation supported by a sketch, why the rate enhancement $\gamma_{||}/\gamma_0$ vanishes as the emitter-mirror distance goes to zero.

(f) (8 pts.) Calculate the decay rate enhancement $\gamma_{\perp}/\gamma_0$ as a function of emitter-mirror distance, where $\gamma_{\perp}$ is the decay rate of a quantum emitter oriented along the z-axis and γ_0 is the emitter's decay rate in vacuum. Add a plot of your result to your plot from problem (d). Give an intuitive explanation, including a sketch, why the rate enhancement goes to 2 as the emitter-mirror distance goes to zero.

Thus far, we have determined the decay rate of a source by considering the work done on a classical dipole by its own field. An alternative approach to the present problem would be to determine the power radiated into the far-field by the dipole. We exemplify this approach here for the dipole oriented perpendicularly to the mirror.

(g) (8 pts.) From the Green function Eq. (1), formulate the far-field Green function. Neglect the presence of the mirror for the moment and determine the electric far-field $\boldsymbol{E}_\infty(\boldsymbol{r})$ in cartesian coordinates generated by a source at $\boldsymbol{r}_0 = (0, 0, d)^T$ at an observation point $\boldsymbol{r} = (x, y, z)^T$.

(h) (6 pts.) Describe in a few words the essential steps of the Fraunhofer approximation regarding phase and amplitude of a field. Apply the Fraunhofer approximation to express the field $\boldsymbol{E}_\infty(\boldsymbol{r})$ derived in the previous problem through the far-field $\boldsymbol{E}_\infty^{(d=0)}(\boldsymbol{r})$ generated by a source located at the origin.

(i) (6 pts.) We now return to the perfect mirror as a background system. Calculate the far-field generated by the mirror dipole located at $\boldsymbol{r}_m$ at the observation point $\boldsymbol{r}$ in front of the mirror in the Fraunhofer approximation. Furthermore, formulate the total far-field generated by the dipole in front of the mirror in the Fraunhofer approximation and express it via $\boldsymbol{E}_\infty^{(d=0)}(\boldsymbol{r})$.

(j) (6 pts.) Calculate the far-field time averaged Poynting vector $\boldsymbol{S}(r,\theta)$ and express it using the power radiated by a dipole in a homogeneous medium P_0, k, and d. Let θ be the polar angle relative to the z-axis.

(k) (6 pts.) Determine the power dissipated by the dipole in front of the mirror by integrating the Poynting vector. From your result, determine the decay rate enhancement of the dipole as a function of its distance to the mirror and show that it is consistent with your result from problem (f).
Hint: The following integral may be helpful

$$\begin{aligned}&\int \mathrm{d}\theta\ \sin^3\theta \cos^2(a\cos\theta)\\ =&\frac{\cos(\theta)\cos(2a\cos(\theta))}{4a^2}+\frac{\left(a^2\cos(2\theta)-a^2-1\right)\sin(2a\cos(\theta))}{8a^3}+\frac{1}{24}(\cos(3\theta)-9\cos(\theta))\end{aligned} \tag{2}$$

(l) (2 pts.) Describe which condition the background system has to fulfill in order for it to be possible to retrieve the decay rate enhancement via the power radiated into the far-field.

2 The quantum efficiency and engineering source brightness [20 pts.]

Engineering the local density of optical states is key when it comes to the development of efficient light sources. In this example, put yourself in the shoes of an engineer working for a company specializing in semiconductor light emitting diodes to understand how you can apply your knowledge about quantum emitters and the LDOS to maximize an LED's efficiency.

Assume we are dealing with a light emitting diode where the quantum emitters are excitons, i.e., electron-hole pairs in a semiconducting material. In most quantum emitters, the excited state can decay not only radiatively but there also exist non-radiative decay channels, whose rates are independent of the photonic environment. Assume that in our LED the excitons recombine radiatively at a rate γ_{rad} and non-radiatively at a rate γ_{nr}.

(a) (2 pts.) Make a sketch of the level scheme and indicate transition rates between levels. What is the total decay rate of the excitons in the LED?

(b) (2 pts.) What is the total decay rate of the excitons in the LED given that we embed the active region in a photonic system that provides an enhancement of the local density of optical states of $\mathcal{A}$ at the emitter position?

(c) (4 pts.) The quantum efficiency QE is defined as the ratio of radiative decay events to the total number of decay events. Show that the quantum efficiency scales linearly with the LDOS enhancement for a source of initially low quantum efficiency.

(d) (4 pts.) Assume you have a single quantum emitter with unit quantum efficiency in a system with LDOS enhancement factor $\mathcal{A}$. What is the maximum photon rate you can get out of this emitter?

(e) (8 pts.) Assume that miniaturization limits the number of quantum emitters on your light emitting device to N. An electrical current pumps emitters in the ground state to the excited state at a rate k_{pump} and excited emitters decay with unit quantum efficiency to the ground state at a rate $\mathcal{A}\gamma$. Calculate the photon production rate of your light emitting device as a function of pump rate k_{pump}. To this end, make use of the fact that in steady state the number of excitation events equals the number of decay events per unit time. Sketch your result and discuss its salient features, especially in the limits of weak and strong pumping. Rescale your axes to suitable dimensionless quantities.