Homework 1

Nano Optics, Fall Semester 2018
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Due date: Friday 12 October 2018; 10:00 a.m.

The goal of this homework is to establish the notation and refresh your knowledge of electrodynamics required to follow the course “Nano Optics”. Use the lecture notes of the course “Electromagnetic Fields and Waves”, available on the website of the Photonics Laboratory, to work through the following problems.
Please stick to the following points:

- Hand in your solution on time (beginning of lecture) in legible hand-written form or typeset using LaTeX (do not provide any code of any programming language, do not send your solutions by email).
- Keep your solutions in the order of the problem set.
- Show explicitly how you arrived at your answer and comment on your thoughts. Focus on the presentation of your solutions as if you were to explain them to a reader who is not familiar with the topic. Be brief but to the point.
- If you decide to generate graphs with a computer (which is encouraged), please insert them at the appropriate position within your solution.
- Always label graphs and axes and provide units where appropriate.
- You can receive a maximum of 100 points for the correct solutions to the problems. For the quality of your presentation, you can receive an additional 10 points.

You are strongly encouraged to discuss the problems with your peers. In case of remaining questions, feel free to approach the course instructors.

1 Maxwell's equations, plane waves and monochromatic fields [optional]

We start with Maxwell's equations for the real valued fields as a function of space and time

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r},t) = \rho(\boldsymbol{r},t), \qquad \nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\partial_t \boldsymbol{B}(\boldsymbol{r},t), \tag{1a}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0, \qquad \nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \partial_t \boldsymbol{D}(\boldsymbol{r},t) + \boldsymbol{j}(\boldsymbol{r},t), \tag{1b}$$

together with the constitutive relations

$$\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t), \qquad \boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \left[\boldsymbol{H}(\boldsymbol{r},t) + \boldsymbol{M}(\boldsymbol{r},t)\right]. \tag{2}$$

(a) (5 ♡) What do the symbols in Eqs. (1) and (2) mean and what are their names? Starting with the charge density $[\rho] = \mathrm{C/m^3}$ and the vacuum permittivity $[\varepsilon_0] = \mathrm{A^2 s^4/(kg\,m^3)}$, derive the SI units of all these quantities.

(b) (6 ♡) Derive the wave equation for the electric field $\boldsymbol{E}(\boldsymbol{r},t)$ in vacuum in the absence of sources, which reads

$$\Box \boldsymbol{E}(\boldsymbol{r},t) = 0, \tag{3}$$

with the d'Alembert operator $\Box = \nabla^2 - \frac{1}{c^2}\partial_t^2$. Which relation do you find for the speed of light c? Derive the wave equation for the magnetic field.

(c) (5 ♡) We now consider a monochromatic field at frequency ω, which we write as $\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left[\boldsymbol{E}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}\right]$. Show that the plane wave

$$\boldsymbol{E}(\boldsymbol{r},t) = \mathrm{Re}\left[\boldsymbol{E}_0\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\boldsymbol{r}-\omega t)}\right] \tag{4}$$

fulfills the wave equation (3). What relation do you find for $|\boldsymbol{k}|$? How does k relate to the wavelength λ?

For monochromatic fields, Maxwell's equations for the complex spatial amplitudes read

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho(\boldsymbol{r}), \qquad \nabla \times \boldsymbol{E}(\boldsymbol{r}) = \mathrm{i}\omega \boldsymbol{B}(\boldsymbol{r}), \tag{5a}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0, \qquad \nabla \times \boldsymbol{H}(\boldsymbol{r}) = -\mathrm{i}\omega \boldsymbol{D}(\boldsymbol{r}) + \boldsymbol{j}(\boldsymbol{r}). \tag{5b}$$

We consider linear isotropic non-chiral media that are described by the constitutive relations

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \varepsilon(\omega) \boldsymbol{E}(\boldsymbol{r}), \qquad \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \mu(\omega) \boldsymbol{H}(\boldsymbol{r}). \tag{6a}$$

(d) (6 ♡) Derive the Helmholtz equation for the electric field assuming no free sources or currents

$$(\nabla^2 + k^2)\boldsymbol{E}(\boldsymbol{r}) = 0 \tag{7}$$

and express the wavenumber k as a function of the refractive index $n(\omega)$. Express the refractive index as a function of the material parameters $\varepsilon(\omega)$ and $\mu(\omega)$.
Derive the Helmholtz equation for the magnetic field.

(e) (3 ♡) Show that the spatial amplitude function of the plane wave in Eq. (4) fulfills the Helmholtz equation.

2 Power flux and intensity [20 pts.]

Any detector at optical frequencies, for example the human eye, only measures the time averaged power flux of an electromagnetic field. The power flux carried by an electromagnetic field is given by the Poynting vector

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t). \tag{8}$$

(a) (2 pts.) Using your results from problem 1(a), determine the SI-units of the Poynting vector. Do they make sense?

(b) (3 pts.) Show that the cycle averaged power flux for a monochromatic field is given by

$$\langle \boldsymbol{S}(\boldsymbol{r}) \rangle = \frac{1}{2}\mathrm{Re}\left[\boldsymbol{E}(\boldsymbol{r}) \times \boldsymbol{H}^*(\boldsymbol{r})\right], \tag{9}$$

where the asterisk denotes the complex conjugate.

The intensity in vacuum is defined as

$$I(\boldsymbol{r}) = \sqrt{\frac{\varepsilon_0}{\mu_0}} \langle \boldsymbol{E}(\boldsymbol{r},t) \cdot \boldsymbol{E}(\boldsymbol{r},t) \rangle \tag{10}$$

where the angled brackets denote the time average over a period much longer than an optical cycle.

(c) (3 pts.) Show that for a monochromatic field the intensity can be expressed by the complex spatial amplitude function as

$$I(\boldsymbol{r}) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} |\boldsymbol{E}(\boldsymbol{r})|^2. \tag{11}$$

(d) (4 pts.) Consider an evanescent wave of the form

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_0 \mathrm{e}^{-\kappa z} \mathrm{e}^{\mathrm{i}k_x x}, \tag{12}$$

with $\kappa, k_x > 0$. Use a small sketch and a few brief sentences to describe how you could generate such an evanescent wave.

(e) (3 pts.) We consider the polarization of the evanescent wave to be $\boldsymbol{E}_0 = [E_0^x, 0, E_0^z]$. Is the field s- or p-polarized? Which relation needs to hold between E_0^x and E_0^z in order for the field to satisfy Maxwell's equations? What is the phase difference between the x- and the z-component of the fields?

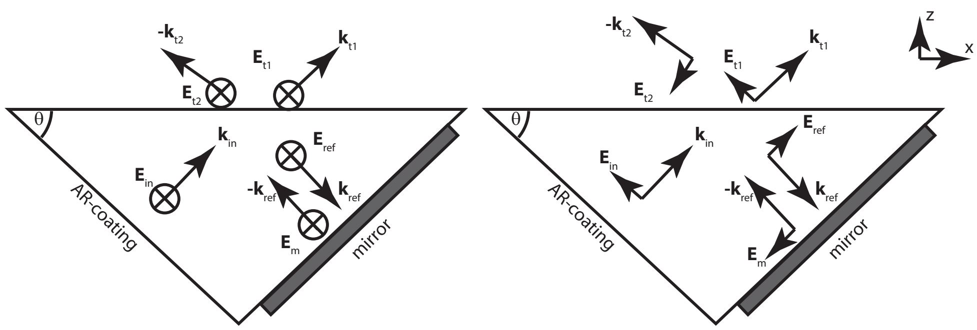
(f) (2 pts.) Calculate the magnetic field of the evanescent wave.

(g) (3 pts.) Calculate the time averaged Poynting vector of the evanescent wave. Comment in a few words on the direction of the energy flux of the evanescent wave.

3 Lithography with evanescent waves [60 pts.]

Most large scale fabrication processes in micro- and nanoelectronics rely on optical lithography methods, where images of the desired structures are projected onto photosensitive polymer masks. The resolution achievable with this technology is limited by diffraction. Here, we investigate the idea of exploiting evanescent waves to generate intensity patterns of minimal spatial periodicity.

We consider an s-polarized plane wave of amplitude E_0 and frequency ω propagating in the xz-plane inside a glass prism with refractive index $n = \sqrt{\epsilon}$, $\mu = 1$, surrounded by air ($n_{\mathrm{air}} = 1$). The surface of the prism, through which the wave enters under normal incidence, is coated with an antireflection coating, such that no reflections happen at that interface. The angle of the prism is θ.



(a) (3 pts.) Formulate the incoming complex field $\boldsymbol{E}_{\mathrm{in}}$ inside the prism. Express the wave vector of the incoming field $\boldsymbol{k}_{\mathrm{in}}$ using the refractive index n, the frequency ω, and the speed of light in vacuum c, as well as the prism angle θ.

(b) (3 pts.) Formulate the complex field reflected at the prism surface $\boldsymbol{E}_{\mathrm{ref}}$ using the Fresnel reflection coefficients. Express the wavevector of the reflected wave by ω, c and n.

(c) (4 pts.) The exit face of the prism is coated with a perfect mirror, such that the wave reflected at the upper surface is reflected back. Formulate the complex field reflected by the mirror $\boldsymbol{E}_{\mathrm{m}}$. Express the phases which are picked up by propagation to the mirror and back and by the reflection by an overall phase ϕ.

(d) (4 pts.) We now turn our attention to the field on the outside of the prism surface. It consists of a transmitted part of the incoming field $\boldsymbol{E}_{\mathrm{in}}$, as well as a transmitted part of the field reflected by the mirror $\boldsymbol{E}_{\mathrm{m}}$. Formulate the complex field stemming from the transmitted part of the incoming primary field using the Fresnel transmission coefficient. Express the wave vector of the transmitted wave $\boldsymbol{k}_{\mathrm{t1}}$ by ω, c and θ.

(e) (4 pts.) Formulate the complex field stemming from the transmission of the field reflected by the mirror surface $\boldsymbol{E}_{\mathrm{m}}$.

(f) (7 pts.) Formulate the total field in the air on the top surface of the prism under the assumption that the prism angle is large enough to generate total internal reflection.
Hint: It may be helpful to express the Fresnel reflection coefficient in polar form $r = |r| \exp[-\mathrm{i}\alpha]$. There is no need for you to worry about the phase angle α. What about $|r|$ in the considered case?

(g) (6 pts.) Show that the field intensity at the surface of the prism reads

$$I^{(s)} = \frac{8\,|E_0|^2}{Z_0} \frac{\cos^2\theta}{1 - 1/n^2} \mathrm{e}^{-2n\frac{\omega}{c}\sqrt{\sin^2\theta - 1/n^2}\,z} \cos^2\left(k_x x + \frac{\alpha - \phi}{2}\right). \tag{13}$$

(h) (6 pts.) We have generated a standing evanescent wave by superposition of two evanescent waves. Compare the periodicity of the field intensity in the air along the surface of the prism with that of two interfering plane waves in air, propagating towards each other under the angles $\pm\theta$. Which price do you pay for the achieved reduction in periodicity?

Thus far, we generated the standing evanescent wave by superposition of two s-polarized fields. From now on, we turn to the case of the incoming field being p-polarized. The prism angle is still θ and the incoming field enters the prism under normal incidence through a non-reflecting interface.

(i) (6 pts.) Formulate the incoming field for a p-polarized plane wave propagating inside the glass prism. Formulate the field reflected at the upper interface. Express the field vector without using the angle θ, instead using the wave vector components k_x and $k_{\mathrm{z,in}}$ and the wavenumber k_{in}.

(j) (5 pts.) Formulate the complex field reflected by the mirror-coated surface of the prism $\boldsymbol{E}_{\mathrm{m}}$ using the phase ϕ from problem (c).

(k) (7 pts.) Formulate the total field on the top surface of the prism assuming total internal reflection by superposing both involved partial fields using the Fresnel transmission coefficients.
Hint: The field is of the form

$$\boldsymbol{E}_{\mathrm{t1}} + \boldsymbol{E}_{\mathrm{t2}} \propto A\mathrm{e}^{-a\zeta z}\mathrm{e}^{\mathrm{i}\delta} \begin{pmatrix} -\zeta\cos(\xi) \\ 0 \\ B\sin(\xi) \end{pmatrix}. \tag{14}$$

with the dependencies $A(t^{(p)}, E_0, n)$, $a(n, \omega, c)$, $\zeta(n, \theta)$, $\delta(\phi, \alpha)$ [here, α is the complex phase angle of the Fresnel reflection coefficient under total internal reflection], $\xi(k_x, x, \delta)$ and $B(\theta)$.

(l) (5 pts.) What is the polarization state of the evanescent field on the prism surface? Are there points where the field is linearly polarized parallel to the prism surface? Are there points where the field is linearly polarized vertically relative to the surface?

4 Far-field imaging [20 pts.]

In this exercise, we acquaint ourselves with the basics of the far-field imaging process. To this end, we consider systems that form an image in the image plane of a sample located in the source plane. These images are formed using lenses, which we model in the thin-lens approximation throughout this exercise. In case you feel like you are missing some background information, may find it helpful to consider any basic physics textbook or a more advanced text on optics, for example the book "Optics" by E. Hecht. Another great resource for practical considerations are the websites of microscope manufacturers, e.g., Nikon, Zeiss, and Leica.
To first order, we can understand the imaging properties of a lens in a ray-optics picture. A simple way to construct the image created by a lens is to make use of the following facts.

1. Light rays passing the center of the lens are not influenced by the lens.
2. Light rays passing through the focal point on one side of the lens travel parallel to the optical axis on the other side of the lens.

(a) (5 pts.) Consider an imaging system composed of a single lens. Using geometric considerations and a sketch, derive the lens equation $1/f = 1/b + 1/g$ with f the focal length, g the distance between lens and object, and b the distance between lens and image.

(b) (4 pts.) For an imaging system based on a single lens, express the magnification m using the focal length f and the object distance g.

In practise, one usually constructs a microscope from (at least) two lenses. The "lens" close to the specimen is called the microscope objective. It is typically an assembly of several optical elements to compensate optimally for abberation effects which we will not consider here. We therefore treat the objective as a simple thin lens, fully characterized by its focal length f_{obj}. The second lens, typically referred to as the "tube lens", is located close to the screen, or a camera, and is characterized by its focal length f_{TL}.

In the lab, we most often encounter "infinity" microscopes. In an infinity imaging system in the ray optics picture, all rays coming from a certain point in the source plane are parallel between the objective and the tube lens.

(c) (3 pts.)In an infinity microscope set up with two lenses with focal lengths f_{TL} and f_{obj}, what is the distance between the sample plane and the objective and what is the distance between the tube lens and the image plane?

(d) (3 pts.) Derive the magnification m of the infinity imaging system given the parameters f_{TL} and f_{obj}.

(e) (3 pts.) Make a ray optics sketch to illustrate how two distinct points in the source plane are imaged into the image plane.

(f) (2 pts.) When you buy a microscope objective, it will be specified with a magnification value. Assume you bought a $100\times$ infinity-corrected objective from your favorite manufacturer (examples are Leica, Nikon, Olympus, Zeiss) and you are operating it with a tube lens of focal length $f_{\text{TL}} = 500$ mm. What is the magnification of your imaging system? Use the internet as a resource to answer this question.