Chapter 2

The Wave Equation

After substituting the fields $\mathbf{D}$ and $\mathbf{B}$ in Maxwell's *curl* equations by the expressions in (1.19), taking their rotation, and combining the two resulting equations we obtain the inhomogeneous wave equations

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial}{\partial t}\left(\mathbf{j}_0 + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}\right) \tag{2.1}$$

$$\nabla \times \nabla \times \mathbf{H} + \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla \times \mathbf{j}_0 + \nabla \times \frac{\partial \mathbf{P}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{M}}{\partial t^2} \tag{2.2}$$

where we have skipped the arguments $(\mathbf{r}, t)$ for simplicity. The expression in the round brackets corresponds to the *total current density*

$$\mathbf{j} = \mathbf{j}_0 + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \ , \tag{2.3}$$

where $\partial \mathbf{P}/\partial t$ is the polarization current density, $\nabla \times \mathbf{M}$ the magnetization current density, and $\mathbf{j}_0$ accounts for the primary (source) current density and any contributions not accounted for by $\partial \mathbf{P}/\partial t$ and $\nabla \times \mathbf{M}$. The wave equations as stated in Eqs. (2.1) and (2.2) do not impose any conditions on the media and hence are generally valid.

2.1 Homogeneous Solution in Free Space

We first consider the solution of the wave equations in free space, in absence of matter and sources. For this case the right hand sides of the wave equations are zero. The operation $\nabla\times\nabla\times$ can be replaced by the identity (1.26), and since in free space $\nabla\cdot\mathbf{E}=0$ the wave equation for $\mathbf{E}$ becomes

$$\boxed{\nabla^2\mathbf{E}(\mathbf{r},t) \;-\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) \;=\; 0} \tag{2.4}$$

with an identical equation for the $\mathbf{H}$-field. Each equation defines three independent scalar equations, namely one for E_x, one for E_y, and one for E_z.

In the one-dimensional scalar case, that is $E(x,t)$, Eq. (2.4) is readily solved by the ansatz of d'Alembert $E(x,t)=E(x-ct)$, which shows that the field propagates through space at the constant velocity c. To tackle three-dimensional vectorial fields we proceed with standard separation of variables

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathbf{R}(\mathbf{r})\,T(t)\,. \tag{2.5}$$

Inserting Eq. (2.5) into Eq. (2.4) leads to[1]

$$c^2\,\frac{\nabla^2\mathbf{R}(\mathbf{r})}{\mathbf{R}(\mathbf{r})} \;-\; \frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial\, t^2} \;=\; 0\,. \tag{2.6}$$

The first term depends only on spatial coordinates $\mathbf{r}$ whereas the second one depends only on time t. Both terms have to add to zero, independent of the values of $\mathbf{r}$ and t. This is only possible if each term is constant. We will denote this constant as $-\omega^2$. The equations for $T(t)$ and $\mathbf{R}(\mathbf{r})$ become

$$\frac{\partial^2}{\partial\, t^2}T(t) + \omega^2\,T(t) \;=\; 0 \tag{2.7}$$

$$\nabla^2\mathbf{R}(\mathbf{r}) + \frac{\omega^2}{c^2}\,\mathbf{R}(\mathbf{r}) \;=\; 0\,. \tag{2.8}$$

Note that both $\mathbf{R}(\mathbf{r})$ and $T(t)$ are real functions of real variables.

[1] The division by $\mathbf{R}$ is to be understood as a multiplication by $\mathbf{R}^{-1}$.

Eq. (2.7) is a harmonic differential equation with the solutions

$$T(t) \;=\; c'_\omega \cos[\omega t] + c''_\omega \sin[\omega t] \;=\; \mathrm{Re}\{c_\omega \exp[-\mathrm{i}\omega t]\}\;, \tag{2.9}$$

where c'_ω and c''_ω are real constants and $c_\omega = c'_\omega + \mathrm{i}c''_\omega$ is a complex constant. Thus, according to ansatz (2.5) we find the solutions

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathbf{R}(\mathbf{r})\,\mathrm{Re}\{c_\omega \exp[-\mathrm{i}\omega t]\} \;=\; \mathrm{Re}\{c_\omega \mathbf{R}(\mathbf{r})\,\exp[-\mathrm{i}\omega t]\}\,. \tag{2.10}$$

In what follows, we will denote $c_\omega R(\mathbf{r})$ as the *complex field amplitude* and abbreviate it by $\mathbf{E}(\mathbf{r})$. Thus, we find

$$\boxed{\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\{\underline{\mathbf{E}}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega t}\}} \tag{2.11}$$

Notice that $\underline{\mathbf{E}}(\mathbf{r})$ is a *complex* field whereas the true field $\mathbf{E}(\mathbf{r},t)$ is real. $\underline{\mathbf{E}}(\mathbf{r})$ is time independent whereas $\mathbf{E}(\mathbf{r},t)$ explicitly depends on time. The underline in $\underline{\mathbf{E}}(\mathbf{r})$ is commonly dropped in the literature, that is, the same symbol $\mathbf{E}$ is used for both the real time-dependent field and the complex spatial part of the field. Here, we will use the underline explicitly in order to clearly differentiate between the real field $\mathbf{E}(\mathbf{r},t)$ and the auxiliary complex field $\underline{\mathbf{E}}(\mathbf{r})$.

Equation (2.11) describes the solution of a *time-harmonic* electric field, a field that oscillates in time at the fixed angular frequency ω. Such a field is also referred to as *monochromatic* field.

After inserting Eq. (2.11) into Eq. (2.4) we obtain

$$\boxed{\nabla^2 \underline{\mathbf{E}}(\mathbf{r}) \;+\; k^2\, \underline{\mathbf{E}}(\mathbf{r}) \;=\; 0} \tag{2.12}$$

with $k = \omega/c$. This equation is referred to as *Helmholtz equation*.

2.1.1 Plane Waves

To solve for the solutions of the Helmholtz equation (2.12) we use the ansatz

$$\underline{\mathbf{E}}(\mathbf{r}) \;=\; \underline{\mathbf{E}}_0\,\mathrm{e}^{\pm\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \;=\; \underline{\mathbf{E}}_0\,\mathrm{e}^{\pm\mathrm{i}(k_x x + k_y y + k_z z)} \tag{2.13}$$

which, after inserting into (2.12), yields

$$k_x^2 \,+\, k_y^2 \,+\, k_z^2 \;=\; \frac{\omega^2}{c^2} \tag{2.14}$$

The left hand side can also be represented by $\mathbf{k}\cdot\mathbf{k} = k^2$, with $\mathbf{k} = (k_x, k_y, k_z)$ being the *wave vector*.

For the following we assume that k_x, k_y, and k_z are real. After inserting Eq. (2.13) into Eq. (2.11) we find the solutions

$$\mathbf{E}(\mathbf{r}, t) \;=\; \mathrm{Re}\{\underline{\mathbf{E}}_0\, \mathrm{e}^{\pm \mathrm{i}\mathbf{k}\cdot\mathbf{r} - i\omega t}\} \tag{2.15}$$

which are called *plane waves* or homogeneous waves. Solutions with the $+$ sign in the exponent are waves that propagate in direction of $\mathbf{k} = [k_x,\, k_y,\, k_z]$. They are denoted *outgoing waves*. On the other hand, solutions with the minus sign are incoming waves and propagate against the direction of $\mathbf{k}$.

Although the field $\mathbf{E}(\mathbf{r}, t)$ fulfills the wave equation it is not yet a rigorous solution of Maxwell's equations. We still have to require that the fields are divergence free, i.e. $\nabla\cdot\mathbf{E}(\mathbf{r}, t) \;=\; 0$. This condition restricts the $\mathbf{k}$-vector to directions perpendicular to the electric field vector ($\mathbf{k}\cdot\underline{\mathbf{E}}_0 \;=\; 0$). Fig. 2.1 illustrates the characteristic features of plane waves.

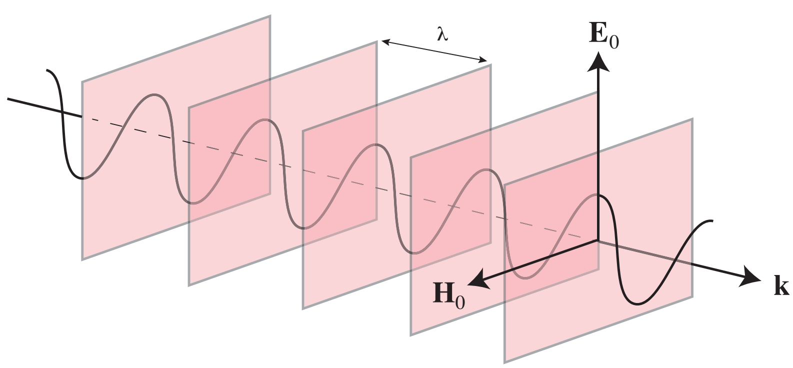

Figure 2.1: Illustration of a plane wave. In free space, the plane wave propagates with velocity c in direction of the wave vector $\mathbf{k} \;=\; (k_x,\, k_y,\, k_z)$. The electric field vector $\mathbf{E}_0$, the magnetic field vector $\mathbf{H}_0$, and $\mathbf{k}$ are perpendicular to each other.

The corresponding magnetic field is readily found by using Maxwell's equation $\nabla \times \underline{\mathbf{E}} = i\omega\mu_0\,\underline{\mathbf{H}}$. We find $\underline{\mathbf{H}}_0 = (\omega\mu_0)^{-1}\,(\mathbf{k} \times \underline{\mathbf{E}}_0)$, that is, the magnetic field vector is perpendicular to the electric field vector and the wavevector $\mathbf{k}$.

Let us consider a plane wave with real amplitude $\mathbf{E}_0$ and propagating in direction of the z axis. This plane wave is represented by $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos[kz - \omega t]$, where $k = |\mathbf{k}| = \omega/c$. If we observe this field at a fixed position z then we'll measure an electric field $\mathbf{E}(t)$ that is oscillating with frequency $f = \omega\,/\,2\pi$. On the other hand, if we take a snapshot of this plane wave at $t = 0$ then we'll observe a field that spatially varies as $\mathbf{E}(\mathbf{r}, t = 0) = \mathbf{E}_0 \cos[k\,z]$. It has a maximum at $z = 0$ and a next maximum at $k\,z = 2\pi$. The separation between maxima is $\lambda = 2\pi/k$ and is called the wavelength. After a time of $t = 2\pi/\omega$ the field reads $\mathbf{E}(\mathbf{r}, t = 2\pi/\omega) = \mathbf{E}_0 \cos[k\,z - 2\pi] = \mathbf{E}_0 \cos[k\,z]$, that is, the wave has propagated a distance of one wavelength in direction of z. Thus, the velocity of the wave is $v_0 = \lambda/(2\pi/\omega) = \omega/k = c$, the vacuum speed of light. For radio waves $\lambda \sim 1\,$km, for microwaves $\lambda \sim 1\,$cm, for infrared radiation $\lambda \sim 10\,\mu$m, for visible light $\lambda \sim 500\,$nm, and for X-rays $\lambda \sim 0.1\,$nm, - the size range of atoms. Figure 2.2 illustrates the length scales associated with the different frequency regions of the

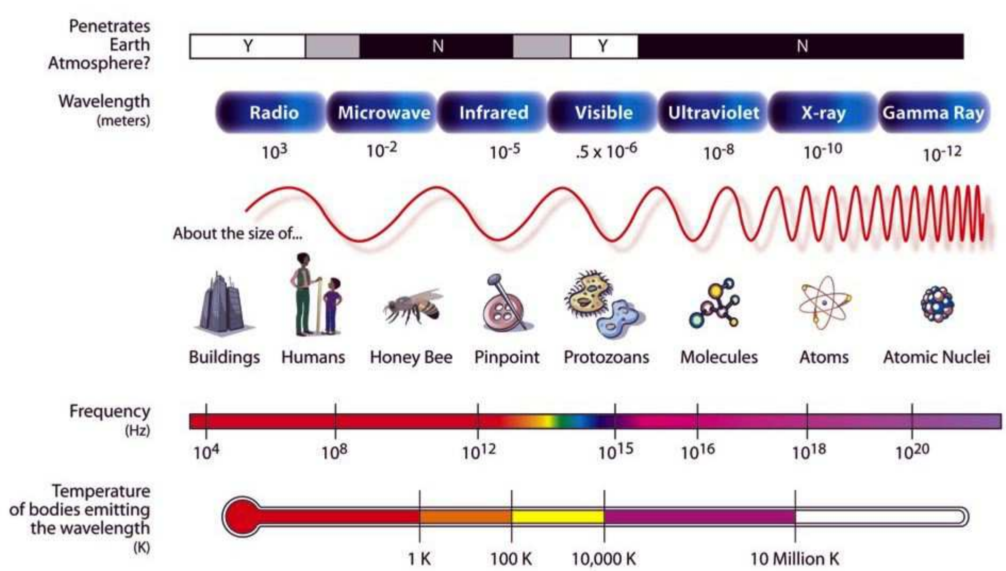

Figure 2.2: Length-scales associated with the different frequency ranges of the electromagnetic spectrum. From mynasadata.larc.nasa.gov.

electromagnetic spectrum.

A plane wave with a fixed direction of the electric field vector $\mathbf{E}_0$ is termed *linearly polarized*. We can form other polarizations states (e.g. circularly polarized waves) by allowing $\mathbf{E}_0$ to rotate as the wave propagates. Such polarization states can be generated by superposition of linearly polarized plane waves.

Plane waves are mathematical constructs that do not exist in practice because their fields $\mathbf{E}$ and $\mathbf{H}$ are infinitely extended in space and therefore carry an infinite amount of energy. Thus, plane waves are mostly used to locally visualize or approximate more complicated fields. They are the simplest form of waves and can be used as a basis to describe other wave fields (angular spectrum representation). For an illustration of plane waves go to http://en.wikipedia.org/wiki/Plane_wave.

2.1.2 Evanescent Waves

So far we have restricted the discussion to real k_x, k_y, and k_z. However, this restriction can be relaxed. For this purpose, let us rewrite the dispersion relation (2.14)

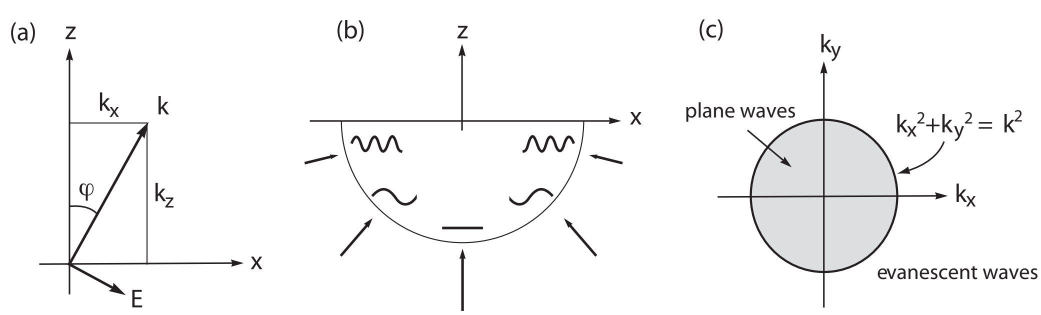

Figure 2.3: (a) Representation of a plane wave propagating at an angle φ to the z axis. (b) Illustration of the transverse spatial frequencies of plane waves incident from different angles. The transverse wavenumber $(k_x^2 + k_y^2)^{1/2}$ depends on the angle of incidence and is limited to the interval $[0 \ldots k]$. (c) The transverse wavenumbers k_x, k_y of plane waves are restricted to a circular area with radius $k = \omega/c$. Evanescent waves fill the space outside the circle.

as

$$k_z \;=\; \sqrt{(\omega^2/c^2) \,-\, (k_x^2 + k_y^2)}\;. \tag{2.16}$$

If we let $(k_x^2 + k_y^2)$ become larger than $k^2 = \omega^2/c^2$ then the square root no longer yields a real value for k_z. Instead, k_z becomes imaginary. The solution (2.15) then turns into

$$\mathbf{E}(\mathbf{r}, t) \;=\; \mathrm{Re}\{\underline{\mathbf{E}}_0\, \mathrm{e}^{\pm \mathrm{i}(k_x x + k_y y) - i\omega t}\}\; \mathrm{e}^{\mp |k_z| z}\;. \tag{2.17}$$

These waves still oscillate like plane waves in the directions of x and y, but they exponentially decay or grow in the direction of z. Typically, they have a plane of origin $z = \mathrm{const.}$ that coincides, for example, with the surface of an insulator or metal. If space is unbounded for $z > 0$ we have to reject the exponentially growing solution on grounds of energy conservation. The remaining solution is exponentially decaying and vanishes at $z \to \infty$ (evanescere = to vanish). Because of their exponential decay, evanescent waves only exist near sources (primary or secondary) of electromagnetic radiation. Evanescent waves form a source of stored energy (reactive power). In light emitting devices, for example, we want to convert evanescent waves into propagating waves to increase the energy efficiency.

To summarize, for a wave with a fixed $(k_x,\, k_y)$ pair we find two different characteristic solutions

$$\begin{aligned} \text{Plane waves :} \quad & \mathrm{e}^{i\,[k_x x \,+\, k_y y]}\, \mathrm{e}^{\pm i\,|k_z| z} \qquad & (k_x^2 + k_y^2 \leq k^2) \\ \text{Evanescent waves :} \quad & \mathrm{e}^{i\,[k_x x \,+\, k_y y]}\, \mathrm{e}^{-|k_z||z|} \qquad & (k_x^2 + k_y^2 > k^2)\,. \end{aligned} \tag{2.18}$$

Plane waves are oscillating functions in z and are restricted by the condition $k_x^2 + k_y^2 \leq k^2$. On the other hand, for $k_x^2 + k_y^2 > k^2$ we encounter evanescent waves with an exponential decay along the z-axis. Figure 2.3 shows that the larger the angle between the $\mathbf{k}$-vector and the z-axis is, the larger the oscillations in the transverse plane will be. A plane wave propagating in the direction of z has no oscillations in the transverse plane ($k_x^2 + k_y^2 = 0$), whereas, in the other limit, a plane wave propagating at a right angle to z shows the highest spatial oscillations in the transverse plane ($k_x^2 + k_y^2 = k^2$). Even higher spatial frequencies are covered by evanescent waves. In principle, an infinite bandwidth of spatial frequencies ($k_x,\, k_y$) can be achieved. However, the higher the spatial frequencies of an evanescent wave are, the faster the fields decay along the z-axis will be. Therefore, practical limitations make the bandwidth finite.

2.1.3 General Homogeneous Solution

A plane or evanescent wave characterized by the wave vector $\mathbf{k} = [k_x,\, k_y,\, k_z]$ and the angular frequency ω is only one of the many homogeneous solutions of the wave equation. To find the general solution we need to sum over all possible plane and evanescent waves, that is, we have to sum waves with all possible $\mathbf{k}$ and ω

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\left\{\sum_{n,m} \underline{\mathbf{E}}_0(\mathbf{k}_n,\omega_m)\, \mathrm{e}^{\pm \mathrm{i}\mathbf{k}_n\cdot\mathbf{r} - i\omega_m t}\right\} \quad \text{with} \quad \mathbf{k}_n\cdot\mathbf{k}_n = \omega_m^2/c^2 \tag{2.19}$$

The condition on the right corresponds to the dispersion relation (2.14). Furthermore, the divergence condition requires that $\underline{\mathbf{E}}_0(\mathbf{k}_n,\omega_m)\cdot\mathbf{k}_n = 0$. We have added the argument $(\mathbf{k}_n,\omega_m)$ to $\underline{\mathbf{E}}_0$ since each plane or evanescent wave in the sum is characterized by a different complex amplitude.

The solution (2.19) assumes that there is a discrete set of frequencies and wavevectors. Such discrete sets can be generated by boundary conditions, for example, in a cavity where the fields on the cavity surface have to vanish. In free space, the sum in Eq. (2.19) becomes continuous and we obtain

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\left\{\int_{\mathbf{k}}\int_{\omega} \underline{\mathbf{E}}_0(\mathbf{k},\omega)\, \mathrm{e}^{\pm \mathrm{i}\mathbf{k}\cdot\mathbf{r} - i\omega t}\, d\omega\, d^3\mathbf{k}\right\} \quad \text{with} \quad \mathbf{k}\cdot\mathbf{k} = \omega^2/c^2 \tag{2.20}$$

which has the appearance of a four-dimensional Fourier transform. Notice that $\underline{\mathbf{E}}_0$ is now a complex amplitude *density*, that is, amplitude per unit ω , unit k_x, unit k_y and unit k_z. The difference between (2.19) and (2.20) is the same as between Fourier series and Fourier transforms.

2.2 Spectral Representation

Let us consider solutions that are represented by a continuous but integrable distribution of frequencies ω, that is, solutions of finite bandwidth. For this purpose we go back to the complex notation of Eq. (2.11) for monochromatic fields and sum (integrate) over all monochromatic solutions

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\left\{\int_{-\infty}^{\infty} \hat{\mathbf{E}}(\mathbf{r},\omega)\, \mathrm{e}^{-\mathrm{i}\omega t}\, d\omega\right\} . \tag{2.21}$$

We have replaced the complex amplitude $\underline{\mathbf{E}}$ by $\hat{\mathbf{E}}$ since we're now dealing with an amplitude per unit frequency, *i.e.* $\hat{\mathbf{E}} = \lim_{\Delta\omega\to 0}[\underline{\mathbf{E}}/\Delta\omega]$. We have also included ω in the argument of $\hat{\mathbf{E}}$ since each solution of constant ω has its own amplitude.

In order to eliminate the 'Re' sign in (2.21) we require that

$$\hat{\mathbf{E}}(\mathbf{r},-\omega) \;=\; \hat{\mathbf{E}}^{*}(\mathbf{r},\omega)\;, \tag{2.22}$$

where * denotes the complex conjugate. This condition leads us to

$$\boxed{\mathbf{E}(\mathbf{r},t) \;=\; \int_{-\infty}^{\infty} \hat{\mathbf{E}}(\mathbf{r},\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega} \tag{2.23}$$

This is simply the Fourier transform of $\hat{\mathbf{E}}$. In other words, $\mathbf{E}(\mathbf{r},t)$ and $\hat{\mathbf{E}}(\mathbf{r},\omega)$ form a time-frequency Fourier transform pair. $\hat{\mathbf{E}}(\mathbf{r},\omega)$ is also denoted as the *temporal spectrum* of $\mathbf{E}(\mathbf{r},t)$. Note that $\hat{\mathbf{E}}$ is generally complex, while $\mathbf{E}$ is always real. The inverse transform reads as

$$\boxed{\hat{\mathbf{E}}(\mathbf{r},\omega) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r},t)\,\mathrm{e}^{i\omega t}\,\mathrm{d}t\;.} \tag{2.24}$$

After applying Fourier transforms to Maxwell's equations (1.31)–(1.34) we obtain

$$\nabla\cdot\hat{\mathbf{D}}(\mathbf{r},\omega) \;=\; \hat{\rho}_0(\mathbf{r},\omega) \tag{2.25}$$

$$\nabla\times\hat{\mathbf{E}}(\mathbf{r},\omega) \;=\; i\omega\hat{\mathbf{B}}(\mathbf{r},\omega) \tag{2.26}$$

$$\nabla\times\hat{\mathbf{H}}(\mathbf{r},\omega) \;=\; -\mathrm{i}\omega\hat{\mathbf{D}}(\mathbf{r},\omega)\;+\;\hat{\mathbf{j}}_0(\mathbf{r},\omega) \tag{2.27}$$

$$\nabla\cdot\hat{\mathbf{B}}(\mathbf{r},\omega) \;=\; 0 \tag{2.28}$$

These equations are the spectral representation of Maxwell's equations. Once a solution for $\hat{\mathbf{E}}$ is found, we obtain the respective time-dependent field by the inverse transform in Eq. (2.23).

2.2.1 Monochromatic Fields

A monochromatic field oscillates at a single frequency ω. According to Eq. (2.11) it can be represented by a complex amplitude $\mathbf{E}(\mathbf{r})$ as

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathrm{Re}\{\underline{\mathbf{E}}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega t}\} \\ &= \mathrm{Re}\{\underline{\mathbf{E}}(\mathbf{r})\}\cos\omega t + \mathrm{Im}\{\underline{\mathbf{E}}(\mathbf{r})\}\sin\omega t \\ &= (1/2)\left[\underline{\mathbf{E}}(\mathbf{r})\,\mathrm{e}^{-\mathrm{i}\omega t} + \underline{\mathbf{E}}^{*}(\mathbf{r})\,\mathrm{e}^{i\omega t}\right] \;. \end{aligned} \tag{2.29}$$

Inserting the last expression into Eq. (2.24) yields the temporal spectrum of a monochromatic wave

$$\hat{\mathbf{E}}(\mathbf{r},\omega') = \frac{1}{2}\left[\underline{\mathbf{E}}(\mathbf{r})\,\delta(\omega'-\omega) + \underline{\mathbf{E}}^{*}(\mathbf{r})\,\delta(\omega'+\omega)\right] \;. \tag{2.30}$$

Here $\delta(x) = \int \exp[\mathrm{i}xt]\,dt/(2\pi)$ is the Dirac delta function. If we use Eq. (2.30) along with similar expressions for the spectra of $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$, ρ_0, and $\mathbf{j}_0$ in Maxwell's equations (2.25)–(2.28) we obtain

$$\nabla\cdot\underline{\mathbf{D}}(\mathbf{r}) = \underline{\rho}_0(\mathbf{r}) \tag{2.31}$$

$$\nabla\times\underline{\mathbf{E}}(\mathbf{r}) = i\omega\underline{\mathbf{B}}(\mathbf{r}) \tag{2.32}$$

$$\nabla\times\underline{\mathbf{H}}(\mathbf{r}) = -\mathrm{i}\omega\underline{\mathbf{D}}(\mathbf{r}) + \underline{\mathbf{j}}_0(\mathbf{r}) \tag{2.33}$$

$$\nabla\cdot\underline{\mathbf{B}}(\mathbf{r}) = 0 \tag{2.34}$$

These equations are used whenever one deals with time-harmonic fields. They are formally identical to the spectral representation of Maxwell's equations (2.25)–(2.28). Once a solution for the complex fields is found, the real time-dependent fields are found through Eq. (2.11).

2.3 Interference of Waves

Detectors do not respond to fields, but to the intensity of fields, which is defined (in free space) as

$$I(\mathbf{r}) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\left\langle \mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)\right\rangle \,, \tag{2.35}$$

with $\langle .. \rangle$ denoting the time-average. For a monochromatic wave, as defined in Eq. (2.29), this expression becomes

$$I(\mathbf{r}) \;=\; \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\;|\underline{\mathbf{E}}(\mathbf{r})|^2\;. \tag{2.36}$$

with $|\underline{\mathbf{E}}|^2 \;=\; \underline{\mathbf{E}}\cdot\underline{\mathbf{E}}^*$. The energy and intensity of electromagnetic waves will be discussed later in Chapter 5. Using Eq. (2.15), the intensity of a plane wave turns out to be $(1/2)(\varepsilon_0/\mu_0)^{1/2}|\underline{\mathbf{E}}_0|^2$ everywhere in space since k_x, k_y, and k_z are all real. For an evanescent wave, however, we obtain

$$I(\mathbf{r}) \;=\; \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}|\underline{\mathbf{E}}_0|^2\,\mathrm{e}^{-2\,k_z\,z}\;, \tag{2.37}$$

that is, the intensity decays exponentially in z-direction. The $1/e$ decay length is $L_z = 1/(2k_z)$ and characterizes the confinement of the evanescent wave.

Next, we take a look at the intensity of a pair of fields

$$\begin{aligned}
I(\mathbf{r}) \;&=\; \sqrt{\varepsilon_0/\mu_0}\;\big\langle[\mathbf{E}_1(\mathbf{r},t)+\mathbf{E}_2(\mathbf{r},t)]\cdot[\mathbf{E}_1(\mathbf{r},t)+\mathbf{E}_2(\mathbf{r},t)]\big\rangle] \\
&=\; \sqrt{\varepsilon_0/\mu_0}\;\big[\big\langle\mathbf{E}_1(\mathbf{r},t)\cdot\mathbf{E}_1(\mathbf{r},t)\big\rangle+\big\langle\mathbf{E}_2(\mathbf{r},t)\cdot\mathbf{E}_2(\mathbf{r},t)\big\rangle+2\big\langle\mathbf{E}_1(\mathbf{r},t)\cdot\mathbf{E}_2(\mathbf{r},t)\big\rangle\big] \\
&=\; I_1(\mathbf{r})\;+\;I_2(\mathbf{r})\;+\;2\,I_{12}(\mathbf{r})
\end{aligned} \tag{2.38}$$

Thus, the intensity of two fields is not simply the sum of their intensities! Instead, there is a third term, a so-called *interference* term. But what about energy conservation? How can the combined power be larger than the sum of the individual power contributions? It turns out that I_{12} can be positive or negative. Furthermore, I_{12} has a directional dependence, that is, there are directions for which I_{12} is positive and other directions for which it is negative. Integrated over all directions, I_{12} cancels and energy conservation is restored.

Coherent Fields

Coherence is a term that refers to how similar two fields are, both in time and space. Coherence theory is a field on its own and we won't dig too deep here. Maximum coherence between two fields is obtained if the fields are monochromatic, their frequencies are the same, and if the two fields have a well-defined phase relationship. Let's have a look at the sum of two plane waves of the *same* frequency ω

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\{\left[\underline{\mathbf{E}}_1\,\mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot\mathbf{r}} + \underline{\mathbf{E}}_2\,\mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\mathbf{r}}\right]\,\mathrm{e}^{-i\omega t}\}\;, \tag{2.39}$$

and let's denote the plane defined by the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ as the (x,y) plane. Then, $\mathbf{k}_1 = (k_{x_1}, k_{y_1}, k_{z_1}) = k(\sin\alpha, \cos\alpha, 0)$ and $\mathbf{k}_2 = (k_{x_2}, k_{y_2}, k_{z_2}) = k(-\sin\beta, \cos\beta, 0)$ (see Figure 2.4). We now evaluate this field along the x-axis and obtain

$$\mathbf{E}(x,t) \;=\; \mathrm{Re}\{\left[\underline{\mathbf{E}}_1\,\mathrm{e}^{\mathrm{i}kx\sin\alpha} + \underline{\mathbf{E}}_2\,\mathrm{e}^{-\mathrm{i}kx\sin\beta}\right]\,\mathrm{e}^{-i\omega t}\}\;, \tag{2.40}$$

which corresponds to the intensity

$$\begin{aligned} I(x) \;&=\; \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left|\left[\underline{\mathbf{E}}_1\,\mathrm{e}^{\mathrm{i}kx\sin\alpha} + \underline{\mathbf{E}}_2\,\mathrm{e}^{-\mathrm{i}kx\sin\beta}\right]\right|^2 \qquad (2.41)\\ &=\; I_1 + I_2 + \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left[\underline{\mathbf{E}}_1\cdot\underline{\mathbf{E}}_2^*\,\mathrm{e}^{\mathrm{i}kx(\sin\alpha+\sin\beta)} \;+\; \underline{\mathbf{E}}_1^*\cdot\underline{\mathbf{E}}_2\,\mathrm{e}^{-\mathrm{i}kx(\sin\alpha+\sin\beta)}\right]\\ &=\; I_1 + I_2 + \sqrt{\varepsilon_0/\mu_0}\;\mathrm{Re}\left\{\underline{\mathbf{E}}_1\cdot\underline{\mathbf{E}}_2^*\,\mathrm{e}^{\mathrm{i}kx(\sin\alpha+\sin\beta)}\right\}\;. \end{aligned}$$

This equation is valid for any complex vectors $\underline{\mathbf{E}}_1$ and $\underline{\mathbf{E}}_2$. We next assume that the two vectors are real and that they are polarized along the z-axis. We then obtain

$$I(x) \;=\; I_1 + I_2 + 2\,\sqrt{I_1\,I_2}\;\cos[kx(\sin\alpha + \sin\beta)]\;. \tag{2.42}$$

The cosine term oscillates between +1 and -1. Therefore, the largest and smallest signals are $I_{min} = (I_1 + I_2) \pm 2\sqrt{I_1\,I_2}$. To quantify the strength of interference one defines the *visibility*

$$\eta \;=\; \frac{I_{max} - I_{min}}{I_{max} + I_{min}}\;, \tag{2.43}$$

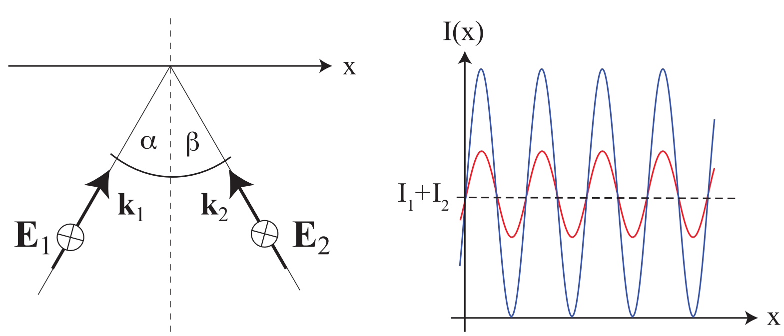

Figure 2.4: Left: Interference of two plane waves incident at angles α and β. Right: Interference pattern along the x-axis for two different visibilities, 0.3 and 1. The average intensity is $I_1 + I_2$.

which has a maximum value of $\eta = 1$ for $I_1 = I_2$. The period of the interference fringes $\Delta x = \lambda/(\sin\alpha + \sin\beta)$ decreases with α and β and is shortest for $\alpha = \beta = \pi/2$, that is, when the two waves propagate head-on against each other. In this case, $\Delta x = \lambda/2$.

Incoherent Fields

Let us now consider a situation for which no interference occurs, namely for two plane waves with different frequencies ω_1 and ω_2. In this case Eq. (2.39) has to be replaced by

$$\mathbf{E}(x,t) \;=\; \mathrm{Re}\{\big[\underline{\mathbf{E}}_1\,\mathrm{e}^{\mathrm{i}k_1 x\sin\alpha - i\omega_1 t} + \underline{\mathbf{E}}_2\,\mathrm{e}^{-\mathrm{i}k_2 x\sin\beta - i\omega_2 t}\big]\}\;. \tag{2.44}$$

Evaluating the intensity yields

$$\begin{aligned} I(x) \;&=\; \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\langle \mathbf{E}(x,t)\cdot\mathbf{E}(x,t)\rangle \qquad (2.45)\\ &=\; I_1 + I_2 + 2\,\sqrt{I_1\,I_2}\;\;\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\,[k_1\sin\alpha + k_2\sin\beta]\,x}\;\left\langle \mathrm{e}^{\mathrm{i}\,(\omega_2-\omega_1)\,t}\right\rangle\right\}\;\;. \end{aligned}$$

The expression $\langle\exp[\mathrm{i}(\omega_2-\omega_1)t]\rangle$ is the time-average of a harmonically oscillating function and yields a result of 0. Therefore, the interference term vanishes and the total intensity becomes

$$I(x) \;=\; I_1 + I_2\;, \tag{2.46}$$

that is, there is no interference.

It has to be emphasized, that the two situations that we analyzed are extreme cases. In practice there is no absolute coherence or incoherence. Any electromagnetic field has a finite line width $\Delta\omega$ that is spread around the center frequency ω. In the case of lasers, $\Delta\omega$ is a few MHz, determined by the spontaneous decay rate of the atoms in the active medium. Thus, an electromagnetic field is at best only partially coherent and its description as a monochromatic field with a single frequency ω is an approximation.