

# Chapter 6

## Radiation

In this chapter we will discuss the emission of electromagnetic radiation from elementary sources. A stationary charge yields a static electric field, but it does not radiate. Similarly, a uniformly moving charge (a current) yields a static magnetic field, but it does not radiate. It is the *acceleration* of charge that gives rise to radiation. The smallest radiating unit is a dipole, an electromagnetic point source. According to linear response theory, a point source excitation yields the *system response function*, from which we can calculate the fields of more complicated sources by using the superposition principle. The system response function is also referred to as the *Green function*.

### 6.1 Green functions

Before calculating the fields radiated by elementary sources let us discuss an important mathematical concept, namely the concept of the Green function. Consider the following general, inhomogeneous equation:

$$\mathcal{L} \mathbf{A}(\mathbf{r}) = \mathbf{B}(\mathbf{r}) . \quad (6.1)$$

$\mathcal{L}$  is a linear operator acting on the vectorfield  $\mathbf{A}$  representing the unknown response of the system. The vectorfield  $\mathbf{B}$  is a known source function and makes the differential equation inhomogeneous. A well-known theorem for linear differential equations states that the general solution is equal to the sum of the complete

homogeneous solution ( $\mathbf{B}=0$ ) and a particular inhomogeneous solution. Here, we assume that the homogeneous solution ( $\mathbf{A}_0$ ) is known. We thus need to solve for an arbitrary particular solution.

Usually it is difficult to find a solution of Eq. (6.1) and it is easier to consider the special inhomogeneity  $\delta(\mathbf{r}-\mathbf{r}')$ , which is zero everywhere, except in the point  $\mathbf{r} = \mathbf{r}'$ . Then, the linear equation reads as

$$\mathcal{L} \mathbf{G}_i(\mathbf{r}, \mathbf{r}') = \mathbf{n}_i \delta(\mathbf{r} - \mathbf{r}') \quad (i = x, y, z), \quad (6.2)$$

where  $\mathbf{n}_i$  denotes an arbitrary constant unit vector. In general, the vectorfield  $\mathbf{G}_i$  is dependent on the location  $\mathbf{r}'$  of the inhomogeneity  $\delta(\mathbf{r}-\mathbf{r}')$ . Therefore, the vector  $\mathbf{r}'$  has been included in the argument of  $\mathbf{G}_i$ . The three equations given by Eq. (6.2) can be written in closed form as

$$\mathcal{L} \overset{\leftrightarrow}{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \overset{\leftrightarrow}{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}'), \quad (6.3)$$

where the operator  $\mathcal{L}$  acts on each column of  $\overset{\leftrightarrow}{\mathbf{G}}$  separately and  $\overset{\leftrightarrow}{\mathbf{I}}$  is the unit tensor. The function  $\overset{\leftrightarrow}{\mathbf{G}}$  fulfilling Eq. (6.3) is known as the dyadic Green function. A dyad is simply a tensor of order two, and is the result of the outer product of two vectors.

In a next step, assume that Eq. (6.3) has been solved and that  $\overset{\leftrightarrow}{\mathbf{G}}$  is known. Postmultiplying Eq. (6.3) with  $\mathbf{B}(\mathbf{r}')$  on both sides and integrating over the volume  $V$  in which  $\mathbf{B} \neq 0$  gives

$$\int_V \mathcal{L} \overset{\leftrightarrow}{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{B}(\mathbf{r}') dV' = \int_V \mathbf{B}(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') dV'. \quad (6.4)$$

The right hand side simply reduces to  $\mathbf{B}(\mathbf{r})$  and with Eq. (6.1) it follows that

$$\mathcal{L} \mathbf{A}(\mathbf{r}) = \int_V \mathcal{L} \overset{\leftrightarrow}{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{B}(\mathbf{r}') dV'. \quad (6.5)$$

If on the right hand side the operator  $\mathcal{L}$  is taken out of the integral, the solution of Eq. (6.1) can be expressed as

$$\mathbf{A}(\mathbf{r}) = \int_V \overset{\leftrightarrow}{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \mathbf{B}(\mathbf{r}') dV'. \quad (6.6)$$

Thus, the solution of the original equation can be found by integrating the product of the dyadic Green function and the inhomogeneity  $\mathbf{B}$  over the source volume  $V$ .

The assumption that the operators  $\mathcal{L}$  and  $\int dV'$  can be interchanged is not strictly valid and special care must be applied if the integrand is not well behaved. Most often  $\vec{G}$  is singular at  $\mathbf{r} = \mathbf{r}'$  and an infinitesimal exclusion volume surrounding  $\mathbf{r} = \mathbf{r}'$  has to be introduced. As long as we consider field points outside of the source volume  $V$ , i.e.  $\mathbf{r} \notin V$ , we do not need to consider these tricky issues.

## 6.2 Scalar and Vector Potentials

The  $\mathbf{E}$  and  $\mathbf{B}$  fields define a total of six functions in space and time. It turns out that these fields are not independent and that one needs fewer functions to uniquely determine the electromagnetic field. The vector potential  $\mathbf{A}$  and the scalar potential  $\phi$  constitute a set of only four functions which, depending on the type of problem, can be reduced to even fewer functions. These potentials are also of key importance in quantum mechanics.

Let's consider Maxwell's equation  $\nabla \cdot \mathbf{B} = 0$  and replace  $\mathbf{B}$  by another function. Because,  $\nabla \cdot \nabla \times = 0$  we choose  $\mathbf{B} = \nabla \times \mathbf{A}$ . Next, we consider Faraday's law  $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$  and replace  $\mathbf{B}$  by  $\nabla \times \mathbf{A}$ . We obtain  $\nabla \times [\mathbf{E} + \partial \mathbf{A} / \partial t] = 0$ . Considering that  $\nabla \times \nabla = 0$ , we set  $[\mathbf{E} + \partial \mathbf{A} / \partial t] = -\nabla \phi$ , which yields  $\mathbf{E} = -\partial \mathbf{A} / \partial t - \nabla \phi$ . To summarize,

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) - \nabla \phi(\mathbf{r}, t) \quad (6.7)$$

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \quad (6.8)$$

It turns out that these definitions of vector potential  $\mathbf{A}$  and scalar potential  $\phi$  are not unique. If the potentials are replaced by new potentials  $\tilde{\mathbf{A}}$ ,  $\tilde{\phi}$  according to

$$\mathbf{A} \rightarrow \tilde{\mathbf{A}} + \nabla \chi \quad \text{and} \quad \phi \rightarrow \tilde{\phi} - \partial \chi / \partial t, \quad (6.9)$$

with  $\chi(\mathbf{r}, t)$  being an arbitrary gauge function, Maxwell's equations remain unaffected. This is easily seen by introducing the above substitutions in the definitions of  $\mathbf{A}$  and  $\phi$ .

### 6.2.1 The Gauges

Any vectorfield  $\mathbf{F}$  is specified by the definition of  $\nabla \cdot \mathbf{F}$  and  $\nabla \times \mathbf{F}$ . A vectorfield with  $\nabla \cdot \mathbf{F} = 0$  is called *transverse*, whereas  $\nabla \times \mathbf{F} = 0$  defines a *longitudinal* field.

So far, we have defined the curl of  $\mathbf{A}$ , i.e.  $\nabla \times \mathbf{A} = \mathbf{B}$ . However, we did not specify  $\nabla \cdot \mathbf{A}$ . The choice of  $\nabla \cdot \mathbf{A}$  does not affect the fields  $\mathbf{E}$  and  $\mathbf{B}$ . Typically one chooses  $\nabla \cdot \mathbf{A}$  such that the wave equation for  $\mathbf{A}$  assumes a simple form or that favorable symmetries can be exploited. To demonstrate this, we consider Maxwell's equation  $\nabla \times \mathbf{H} = \partial \mathbf{D} / \partial t + \mathbf{j}_0$ . Using the relations (1.19) we obtain  $\nabla \times \mathbf{B} - (1/c^2) \partial \mathbf{E} / \partial t = \mu_0 [\nabla \times \mathbf{M} + \partial \mathbf{P} / \partial t + \mathbf{j}_0]$ , where the expression in brackets is the total current density  $\mathbf{j}$ . Inserting Eqs. (6.7) and (6.8) yields

$$\nabla \times \nabla \times \mathbf{A} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} + \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} = \mu_0 \mathbf{j}, \quad (6.10)$$

with  $\mathbf{j} = [\nabla \times \mathbf{M} + \partial \mathbf{P} / \partial t + \mathbf{j}_0]$ , which can be rewritten as

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = -\mu_0 \mathbf{j} + \nabla \left[ \nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right]. \quad (6.11)$$

The expression in brackets contains a  $\nabla \cdot \mathbf{A}$  term, which we can choose as we wish. Finally, we also express Gauss' law  $\nabla \cdot \mathbf{D} = \rho_0$  in terms of  $\mathbf{A}$  and  $\phi$  and obtain

$$\nabla \cdot (\partial \mathbf{A} / \partial t + \nabla \phi) = -\rho / \epsilon_0. \quad (6.12)$$

There is again a  $\nabla \cdot \mathbf{A}$  term that can be arbitrarily chosen.

#### Lorenz Gauge

In the Lorenz gauge one chooses  $\nabla \cdot \mathbf{A} = -(1/c^2) \partial \phi / \partial t$ <sup>1</sup>, which yields

$$\left[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{A} = -\mu_0 \mathbf{j} \quad (6.13)$$

$$\left[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \phi = -\frac{1}{\epsilon_0} \rho \quad (6.14)$$

<sup>1</sup>This has the form of a continuity equation ( $\mathbf{A}$  is the current density and  $\phi/c^2$  is the charge density).

Thus, we obtain two decoupled partial differential equations (wave equations) of the *same* form for  $\mathbf{A}$  and  $\phi$ . Note, that one ends up with the same differential equations by a proper choice of the gauge function  $\chi$  in Eq. (6.9).

The advantage of the Lorenz gauge is that the vectorial differential equation for  $\mathbf{A}$  is decoupled into a set of three independent scalar differential equations, that is, each vector component  $A_i$  depends only on the source component  $j_i$ . There is no mixing of components  $i \in [x, y, z]$ .

### Coulomb Gauge

In the Coulomb gauge one chooses  $\nabla \cdot \mathbf{A} = 0$ . This gauge is also referred to as the transverse gauge or the minimal coupling gauge. With this choice of gauge Eqs. (6.11) and (6.12) reduce to

$$\begin{aligned} \left[ \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \mathbf{A} &= -\mu_0 \mathbf{j} + \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} \\ \nabla^2 \phi &= -\frac{1}{\varepsilon_0} \rho. \end{aligned} \quad (6.15)$$

Here, the scalar potential  $\phi$  is determined by a *Poisson equation*, that is, there is no retardation and  $\phi$  is an instantaneous function. The Coulomb gauge is mostly used for problems in quantum optics and is less important for this course. There are many more gauges, which we won't discuss here. Among them are the Poincaré gauge, the Landau gauge, and the Weyl gauge. We will be mostly dealing with the Lorenz gauge.

Note that by going from  $\mathbf{E}$ ,  $\mathbf{H}$  to  $\mathbf{A}$ ,  $\phi$  we reduced the field parameters from six to four (three per vector and one per scalar). It turns out that the four parameters are still redundant and that they can be reduced even more. One way is to introduce the so-called Hertz potential  $\mathbf{\Pi}(\mathbf{r})$ , which has only three components. The vector and scalar potentials are related to  $\mathbf{\Pi}$  as  $\mathbf{A} = (1/c^2) \partial \mathbf{\Pi} / \partial t$  and  $\phi = -\nabla \cdot \mathbf{\Pi}$ , respectively. Using so-called Debye potentials is yet another representation of fields, but these won't be discussed here.

### 6.3 Dipole Radiation

In this section we will derive the electromagnetic field of a dipole source, the smallest radiating system. Mathematically, the dipole source corresponds to a delta excitation, and the response to a delta excitation is the *Green function* discussed previously. Any source can be thought of as being composed of individual point sources with different origins. In other words, any macroscopic source volume can be chopped up into little infinitesimal cubes, each of which carries a current that is represented by a delta function.

As shown in Fig. 6.1, a dipole is a separation of a pair of charges by an infinitesimal distance  $ds = \mathbf{n}_s ds$ . The *dipole moment*  $\mathbf{p}$  is defined as

$$\mathbf{p}(t) = q(t) ds. \quad (6.16)$$

The time derivative of the dipole moment is

$$\frac{\partial}{\partial t} \mathbf{p}(t) = \left[ \frac{\partial q(t)}{\partial t} \mathbf{n}_s \right] ds = [\mathbf{j}_0 da] ds = \mathbf{j}_0 dV, \quad (6.17)$$

where  $[\mathbf{j}_0 \cdot \mathbf{n}_s] da$  is the current flowing through the cross-sectional area  $da$ . The product of  $da$  and  $ds$  defines the infinitesimal source volume  $dV$ .

Let us now consider an arbitrary macroscopic source current density  $\mathbf{j}_0(\mathbf{r}, t)$  that is entirely contained within the volume  $V$ . We can express this current density in terms of a sum of microscopic point current densities. In terms of the Dirac delta

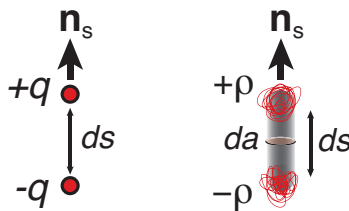


Figure 6.1: Illustration of a dipole with moment  $\mathbf{p} = q ds = q ds \mathbf{n}_s$ . Left: in terms of discrete point charges  $q$ ; Right: in terms of charge distributions  $\rho$ .

function  $\delta$  this sum becomes

$$\mathbf{j}_0(\mathbf{r}, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{j}_0(x', y', z', t) \delta(x - x') \delta(y - y') \delta(z - z') dx' dy' dz' \quad (6.18)$$

$$= \int_V \mathbf{j}_0(\mathbf{r}', t) \delta(\mathbf{r} - \mathbf{r}') dV', \quad (6.19)$$

Here,  $\mathbf{j}_0(\mathbf{r}', t)\delta(\mathbf{r} - \mathbf{r}')dV'$  are elementary point currents. Using Eq. (6.17) we obtain the current density of an elementary dipole source

$$\mathbf{j}_0(\mathbf{r}, t) = \frac{\partial}{\partial t} \mathbf{p}(t) \delta(\mathbf{r} - \mathbf{r}_0) \quad (6.20)$$

where  $\mathbf{r}_0$  is the dipole location.

In principle, we can now determine the fields  $\mathbf{E}$  and  $\mathbf{H}$  radiated by a dipole  $\mathbf{p}$  in free space by inserting (6.20) into the wave equation (2.1). This task is, however, easier accomplished by replacing  $\mathbf{E}$  and  $\mathbf{H}$  by the potentials  $\mathbf{A}$  and  $\phi$ , as discussed previously.

### 6.3.1 Vector Potential of a Time-Harmonic Dipole

To calculate the fields of a dipole we will use the Lorenz gauge. The appeal of the Lorenz gauge is its symmetry, that is, there is a *scalar* wave equation of the form  $[\nabla^2 - (1/c^2)\partial^2/\partial t^2]\Psi = \Theta$  for any of the field components  $A_x, A_y, A_z, \phi$ .

#### Vector and Scalar Potentials with Harmonic Time Dependence

The differential equations for  $\mathbf{A}$  and  $\phi$  in Eqs. (6.13) and (6.14) are valid for arbitrary time dependence. They are expressed in terms of the *total* charge and current densities  $\rho$  and  $\mathbf{j}$ . For time-harmonic fields we set  $\mathbf{A}(\mathbf{r}, t) = \text{Re}\{\underline{\mathbf{A}}(\mathbf{r}) \exp(-i\omega t)\}$  and  $\phi(\mathbf{r}, t) = \text{Re}\{\underline{\phi}(\mathbf{r}) \exp(-i\omega t)\}$  and use the Lorenz gauge in the form

$$\nabla \cdot \underline{\mathbf{A}}(\mathbf{r}) = i(k^2/\omega) \underline{\phi}(\mathbf{r}), \quad (6.21)$$

where  $k^2 = (\omega^2/c^2)\mu(\omega)\epsilon(\omega)$ . This choice allows us to split off the polarization and magnetization current densities from  $\mathbf{j}$  and accommodate them in  $\mu$  and  $\epsilon$ .<sup>2</sup> The equations for the complex potentials  $\underline{\mathbf{A}}(\mathbf{r})$  and  $\underline{\phi}(\mathbf{r})$  become

$$[\nabla^2 + k^2] \underline{\mathbf{A}}(\mathbf{r}) = -\mu_0\mu \underline{\mathbf{j}}_0(\mathbf{r}) \quad (6.22)$$

$$[\nabla^2 + k^2] \underline{\phi}(\mathbf{r}) = -\frac{1}{\epsilon_0\epsilon} \underline{\rho}_0(\mathbf{r}) \quad (6.23)$$

with  $\underline{\rho}_0$  and  $\underline{\mathbf{j}}_0$  being the primary (source) charge and current densities, respectively.

Let us consider a time-harmonic dipole  $\mathbf{p}(t) = \text{Re}\{\underline{\mathbf{p}} \exp[-i\omega t]\}$ , with  $\underline{\mathbf{p}}$  being the complex amplitude. Using complex notation the equations for the components of the vector potential  $\underline{\mathbf{A}}$  become

$$[\nabla^2 + k^2] \underline{A}_i(\mathbf{r}) = i\omega\mu_0\mu \underline{p}_i \delta(\mathbf{r} - \mathbf{r}') \quad (6.24)$$

where we used Eq. (6.20) for the current density of the dipole field. Let us now define the *scalar Green function* as  $G_0 = i \underline{A}_i / (\omega\mu_0\mu \underline{p}_i)$ . Then, Eq. (6.24) turns into

$$[\nabla^2 + k^2] G_0(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \quad (6.25)$$

We have included the origin of the the point source ( $\mathbf{r}'$ ) in the argument of  $G_0$  to remind us where the origin of the point source is. In other words  $G_0(\mathbf{r}, \mathbf{r}')$  is the response at  $\mathbf{r}$  to a dipole source at  $\mathbf{r}'$ .

In free space,  $G_0(\mathbf{r}, \mathbf{r}')$  must be point symmetric, because the fields depend only on the radial distance  $R = |\mathbf{r} - \mathbf{r}'|$  and things don't change if the coordinate system is rotated around the origin  $\mathbf{r} = \mathbf{r}'$ . To solve Eq. (6.25) we will try the following ansatz

$$R G_0 = a_1 e^{ikR} + a_2 e^{-ikR}, \quad (6.26)$$

which is a superposition of an outgoing and an incoming wave. After inserting into Eq. (6.25) and integrating on both sides over a small spherical volume  $\Delta V$  centered at  $R = 0$  and with radius  $r_o$ , we obtain

$$\int_{\Delta V} \nabla^2 \frac{1}{R} dV + k^2 \int_{\Delta V} \frac{1}{R} dV = \frac{1}{a_1 + a_2} \quad (6.27)$$

<sup>2</sup>We assume that in the region of interest the material properties are linear and homogeneous.



The second term integrates to  $2\pi k^2 r_o^2$  and the first term is calculated as

$$\int_{\Delta V} \nabla \cdot \left[ \nabla \frac{1}{R} \right] dV = \int_{\partial \Delta V} \left[ \nabla \frac{1}{R} \right] \cdot \mathbf{n}_R da = - \int_{\partial \Delta V} \frac{\mathbf{n}_R \cdot \mathbf{n}_R}{R^2} da = -4\pi. \quad (6.28)$$

where we used Gauss' theorem (1.27). Thus, for  $r_o \rightarrow 0$  we obtain  $(a_1 + a_2) = 1/4\pi$ . Finally, in free-space the radiation released by the point source is not coming back, which implies that we can drop the incoming wave in Eq. (6.26) or, equivalently, set  $a_2 = 0$ . The solution for the scalar Green function becomes

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (6.29)$$

$G_0$  defines the vector potential at  $\mathbf{r}$  due to a dipole  $\mathbf{p}$  at  $\mathbf{r}'$  according to

$$\underline{\mathbf{A}}(\mathbf{r}) = -i\omega\mu_0\mu \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \underline{\mathbf{p}}. \quad (6.30)$$

What if the source is not a dipole but an arbitrary current distribution? In this case we go back to Eq. (6.25) and multiply both sides with  $\mu_0\mu \underline{j}_{0i}(\mathbf{r}')$ , where  $\underline{j}_{0i}$  is the  $i$ -th vector component of the total current density  $\underline{\mathbf{j}}_0$ . Integrating both sides over the source volume  $V$  yields

$$\begin{aligned} \mu_0\mu \int_V [\nabla^2 + k^2] G_0(\mathbf{r}, \mathbf{r}') \underline{j}_{0i}(\mathbf{r}') dV' &= -\mu_0\mu \int_V \delta(\mathbf{r}-\mathbf{r}') \underline{j}_{0i}(\mathbf{r}') dV' \\ &= -\mu_0\mu \underline{j}_{0i}(\mathbf{r}), \end{aligned} \quad (6.31)$$

where we used the definition of the delta function. We now assume that the observation point  $\mathbf{r}$  is outside the source volume described by the coordinate  $\mathbf{r}'$ . In this case, we can swap the sequence of integration and differentiation in Eq. (6.31) and obtain

$$[\nabla^2 + k^2] \mu_0\mu \int_V G_0(\mathbf{r}, \mathbf{r}') \underline{j}_{0i}(\mathbf{r}') dV' = -\mu_0\mu \underline{j}_{0i}(\mathbf{r}). \quad (6.32)$$

Comparing this equation with Eq. (6.22) we conclude that

$$\underline{\mathbf{A}}(\mathbf{r}) = \mu_0\mu \int_V G_0(\mathbf{r}, \mathbf{r}') \underline{\mathbf{j}}_0(\mathbf{r}') dV' \quad (6.33)$$

Thus, the solution for  $\underline{\mathbf{A}}$  turns out to be the linear superposition of dipole fields with different origins  $\mathbf{r}'$  and different weights  $\underline{\mathbf{j}}_0$ .

### 6.3.2 Electric and Magnetic Dipole Fields

Now that we have derived the vector potential  $\underline{\mathbf{A}}$  of an oscillation dipole, we find the magnetic field using  $\underline{\mathbf{B}} = \nabla \times \underline{\mathbf{A}}$  and the electric field using Maxwell's equation  $\underline{\mathbf{E}} = i(\omega/k^2)\nabla \times \underline{\mathbf{B}}$ . Skipping the details of the calculation, we find

$$\underline{\mathbf{E}}(\mathbf{r}) = \omega^2 \mu_0 \mu \overset{\leftrightarrow}{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}') \underline{\mathbf{p}} \quad (6.34)$$

$$\underline{\mathbf{H}}(\mathbf{r}) = -i\omega \left[ \nabla \times \overset{\leftrightarrow}{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}') \right] \underline{\mathbf{p}} \quad (6.35)$$

where we introduced the so-called *dyadic Green function*  $\overset{\leftrightarrow}{\mathbf{G}}_0$  defined as

$$\overset{\leftrightarrow}{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}') = \left[ \overset{\leftrightarrow}{\mathbf{I}} + \frac{1}{k^2} \nabla \nabla \right] G_0(\mathbf{r}, \mathbf{r}') \quad (6.36)$$

with  $G_0$  being the scalar Green function (6.29) and  $\overset{\leftrightarrow}{\mathbf{I}}$  being the unit tensor. Notice that  $\overset{\leftrightarrow}{\mathbf{G}}_0$  is a tensor. It is straightforward to calculate  $\overset{\leftrightarrow}{\mathbf{G}}_0$  in the major three coordinate systems. In a Cartesian system  $\overset{\leftrightarrow}{\mathbf{G}}_0$  can be written as

$$\overset{\leftrightarrow}{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}') = \frac{\exp(ikR)}{4\pi R} \left[ \left( 1 + \frac{ikR - 1}{k^2 R^2} \right) \overset{\leftrightarrow}{\mathbf{I}} + \frac{3 - 3ikR - k^2 R^2}{k^2 R^2} \frac{\mathbf{R}\mathbf{R}}{R^2} \right] \quad (6.37)$$

where  $R$  is the absolute value of the vector  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$  and  $\mathbf{R}\mathbf{R}$  denotes the outer product of  $\mathbf{R}$  with itself. Equation (6.37) defines a symmetric  $3 \times 3$  matrix

$$\overset{\leftrightarrow}{\mathbf{G}}_0 = \begin{bmatrix} G_{xx} & G_{xy} & G_{xz} \\ G_{xy} & G_{yy} & G_{yz} \\ G_{xz} & G_{yz} & G_{zz} \end{bmatrix}, \quad (6.38)$$

which, together with Eqs. (6.34) and (6.35), determines the electromagnetic field of an arbitrary electric dipole  $\underline{\mathbf{p}}$  with Cartesian components  $p_x, p_y, p_z$ . The tensor  $[\nabla \times \overset{\leftrightarrow}{\mathbf{G}}_0]$  can be expressed as

$$\nabla \times \overset{\leftrightarrow}{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}') = \frac{\exp(ikR)}{4\pi R} \frac{k(\mathbf{R} \times \overset{\leftrightarrow}{\mathbf{I}})}{R} \left( i - \frac{1}{kR} \right), \quad (6.39)$$

where  $\mathbf{R} \times \overset{\leftrightarrow}{\mathbf{I}}$  denotes the matrix generated by the cross-product of  $\mathbf{R}$  with each column vector of  $\overset{\leftrightarrow}{\mathbf{I}}$ .

### Near-fields and Far-fields

The Green function  $\vec{\mathbf{G}}_0$  has terms in  $(kR)^{-1}$ ,  $(kR)^{-2}$  and  $(kR)^{-3}$ . In the *far-field*, for which  $R \gg \lambda$ , only the terms with  $(kR)^{-1}$  survive. On the other hand, the dominant terms in the *near-field*, for which  $R \ll \lambda$ , are the terms with  $(kR)^{-3}$ . The terms with  $(kR)^{-2}$  dominate the *intermediate-field* at  $R \approx \lambda$ . To distinguish these three ranges it is convenient to write

$$\vec{\mathbf{G}}_0 = \vec{\mathbf{G}}_{\text{NF}} + \vec{\mathbf{G}}_{\text{IF}} + \vec{\mathbf{G}}_{\text{FF}}, \quad (6.40)$$

where the near-field ( $G_{\text{NF}}$ ), intermediate-field ( $G_{\text{IF}}$ ) and far-field ( $G_{\text{FF}}$ ) Green functions are given by

$$\vec{\mathbf{G}}_{\text{NF}} = \frac{\exp(ikR)}{4\pi R} \frac{1}{k^2 R^2} [-\vec{\mathbf{I}} + 3\mathbf{R}\mathbf{R}/R^2], \quad (6.41)$$

$$\vec{\mathbf{G}}_{\text{IF}} = \frac{\exp(ikR)}{4\pi R} \frac{i}{kR} [\vec{\mathbf{I}} - 3\mathbf{R}\mathbf{R}/R^2], \quad (6.42)$$

$$\vec{\mathbf{G}}_{\text{FF}} = \frac{\exp(ikR)}{4\pi R} [\vec{\mathbf{I}} - \mathbf{R}\mathbf{R}/R^2]. \quad (6.43)$$

Notice that the intermediate-field is  $90^\circ$  out of phase with respect to the near- and far-field.

Because the dipole is located in a homogeneous environment, all three dipole orientations lead to fields that are identical upon suitable frame rotations. We therefore choose a coordinate system with origin at  $\mathbf{r}' = \mathbf{r}_0$  and a dipole orientation along the z-axis, i.e.  $\underline{\mathbf{p}} = p \mathbf{n}_z$  (see Fig. 6.2). It is most convenient to represent the dipole fields in spherical coordinates  $\mathbf{r} = (r, \vartheta, \varphi)$  and in spherical vector components  $\underline{\mathbf{E}} = (\underline{E}_r, \underline{E}_\vartheta, \underline{E}_\varphi)$ . In this system the field components  $\underline{E}_\varphi$  and  $\underline{H}_r, \underline{H}_\vartheta$  are identical to zero and the only non-vanishing field components are

$$\underline{E}_r = \frac{p \cos \vartheta}{4\pi \varepsilon_0 \varepsilon} \frac{\exp(ikr)}{r} k^2 \left[ \frac{2}{k^2 r^2} - \frac{2i}{kr} \right], \quad (6.44)$$

$$\underline{E}_\vartheta = \frac{p \sin \vartheta}{4\pi \varepsilon_0 \varepsilon} \frac{\exp(ikr)}{r} k^2 \left[ \frac{1}{k^2 r^2} - \frac{i}{kr} - 1 \right], \quad (6.45)$$

$$\underline{H}_\varphi = -\frac{p \sin \vartheta}{4\pi \varepsilon_0 \varepsilon} \frac{\exp(ikr)}{r} k^2 \left[ \frac{i}{kr} + 1 \right] \sqrt{\frac{\varepsilon_0 \varepsilon}{\mu_0 \mu}}. \quad (6.46)$$

The fact that  $\underline{E}_r$  has no far-field term ensures that the far-field is purely transverse. Furthermore, since the magnetic field has no terms in  $(kr)^{-3}$  the near-field

is dominated by the electric field (see Fig. 6.3). This justifies a quasi-electrostatic consideration in the region very close to the source.

### The Phase of the Dipole Field

It is instructive to also have a look at the phase of the dipole field since close to the origin it deviates considerably from the familiar phase of a spherical wave  $\exp[ikr]$ . The phase of the field is defined relative to the oscillation of the dipole  $p_z$ . In Fig. 6.4 we plot the phase of the field  $E_z$  along the  $x$ -axis and along the  $z$ -axis (c.f. Fig. 6.2). Interestingly, at the origin the phase of the transverse field is  $180^\circ$  out of phase with the dipole oscillation (Fig. 6.4(a)). The phase of the transverse field then drops to a minimum value at a distance of  $x \sim \lambda/5$  after which it increases and then asymptotically approaches the phase of a spherical wave with origin at the dipole (dashed line). On the other hand, the phase of the longitudinal field, shown in Fig. 6.4(b), starts out to be the same as for the oscillating dipole, but it runs  $90^\circ$  out of phase for distances  $z \gg \lambda$ . The reason for this behavior is the missing far-field term in the longitudinal field (c.f. Equation (6.44)). The  $90^\circ$  phase shift is due to the intermediate field represented by the Green function in Eq. (6.42). The

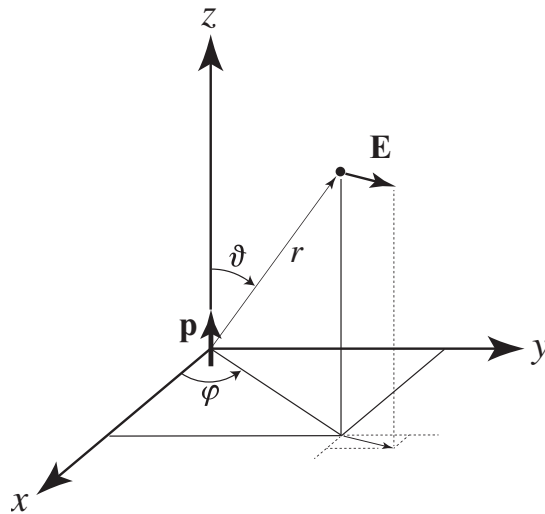


Figure 6.2: The fields of a dipole are most conveniently represented in a spherical coordinate system  $(r, \vartheta, \varphi)$  in which the dipole points along the  $z$ -axis ( $\vartheta = 0$ ).

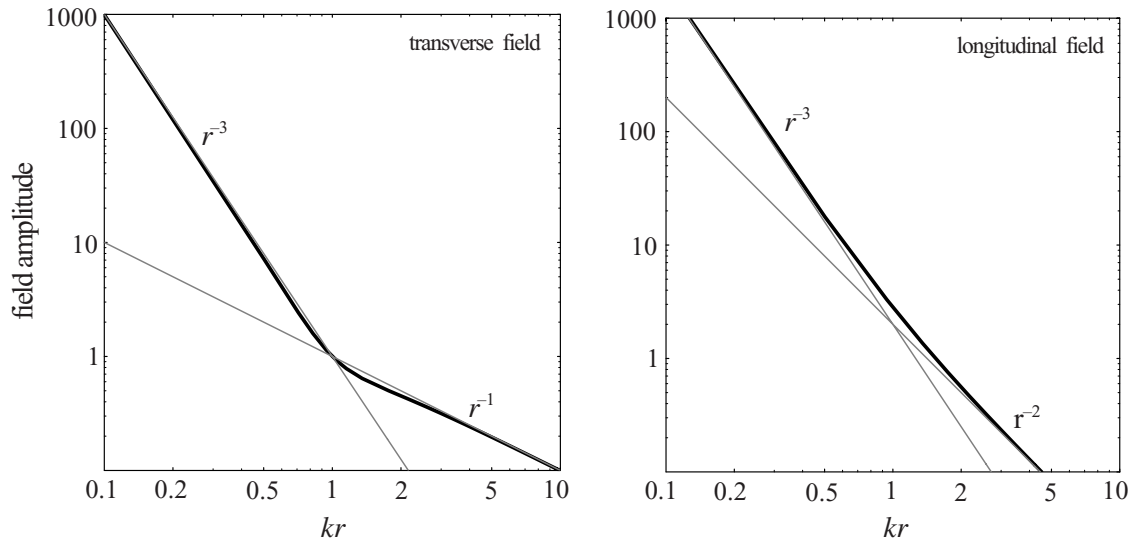


Figure 6.3: Radial decay of the dipole's transverse and longitudinal fields. The curves correspond to the absolute value of the expressions in brackets of Eqs. (6.44) and (6.45), respectively. While both the transverse and the longitudinal field contribute to the near-field, only the transverse field survives in the far-field. Notice that the intermediate-field with  $(kr)^{-2}$  does not really show up for the transverse field. Instead the near-field dominates for  $(kr) < 1$  and the far-field for  $(kr) > 1$ .

same intermediate field is also responsible for the dip near  $x \sim \lambda/5$  in Fig. 6.4(a). This phase dip is of relevance for the design of multi-element antennas, such as the Yagi-Uda antennas. It is important to remember that close to the source the phase of the field *does not* evolve linearly with distance and that the phase can be advanced or delayed by small distance variations.

### 6.3.3 Radiation Patterns and Power Dissipation

To calculate the power radiated by the dipole  $\mathbf{p}$  we consider a fictitious spherical surface  $\partial V$  of radius  $R_o$  centered at the origin of the dipole. According to Poynting's theorem discussed in Section 5.1, the net power  $\bar{P}$  radiated corresponds to the flux of the time-averaged Poynting vector through the enclosing spherical surface (see

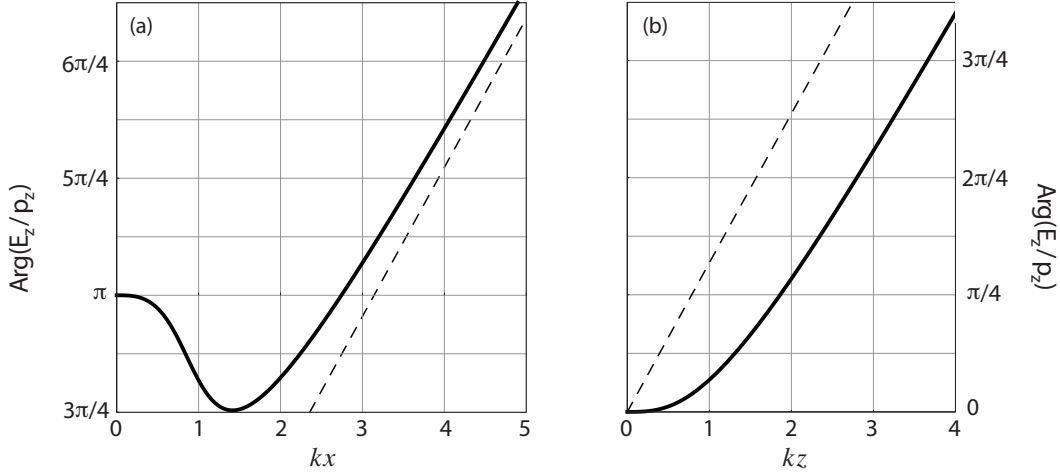


Figure 6.4: Phase of the electric field near the origin. (a) Phase of the transverse field  $E_z$  evaluated along the  $x$ -axis. At the origin, the electric field is  $180^\circ$  out of phase with the dipole. The phase drops to a minimum at a distance of  $x \sim \lambda/5$ . For larger distances, the phase approaches that of a spherical wave  $\exp[ikr]$  (dashed line). (b) Phase of the longitudinal field  $E_z$  evaluated along the  $z$ -axis. At the origin, the electric field is in phase with the dipole. At larger distances, the phase is  $90^\circ$  out of phase with a spherical wave  $\exp[ikr]$  (dashed line).

Eq. 5.9)

$$\bar{P} = \frac{1}{2} \int_{\partial V} \text{Re} \{ \underline{\mathbf{E}}(\mathbf{r}) \times \underline{\mathbf{H}}^*(\mathbf{r}) \} \cdot \mathbf{n} \, da. \quad (6.47)$$

Because we chose a spherical surface, the normal vector  $\mathbf{n}$  is a radial vector and hence we only need to calculate the radial component of  $\langle \mathbf{S} \rangle$ . Using Eqs. (6.45) and (6.46) we find

$$\bar{P} = \frac{1}{2} \int_{\partial V} \text{Re} \{ \underline{E}_\vartheta \underline{H}_\varphi^* \} r^2 \sin \vartheta \, d\vartheta \, d\varphi, \quad (6.48)$$

which yields

$$\bar{P} = \frac{p^2}{4\pi\epsilon_0\epsilon} \frac{n^3\omega^4}{3c^3} = \frac{p^2\omega k^3}{12\pi\epsilon_0\epsilon} \quad (6.49)$$

where  $p = |\underline{\mathbf{p}}|$ . We find that the radiated power scales with the fourth power of the frequency and that only the far-field of the dipole contributes to the net energy transport.

To determine the radiation pattern we calculate the power  $\bar{P}(\vartheta, \varphi)$  radiated into an infinitesimal unit solid angle  $d\Omega = \sin \vartheta d\vartheta d\varphi$  and divide by the total radiated power  $\bar{P}$

$$\frac{\bar{P}(\vartheta, \varphi)}{\bar{P}} = \frac{3}{8\pi} \sin^2 \vartheta . \quad (6.50)$$

Thus, in the far-field most of the energy is radiated perpendicularly to the dipole moment (see Fig. 6.5) and there is no radiation at all in the direction of the dipole.

## 6.4 Dipole Radiation in Arbitrary Environments

So far we have considered a dipole in a homogeneous space characterized by  $\mu$  and  $\varepsilon$ . What happens if we place the dipole near a material boundary or enclose it in a box? Will the dipole still dissipate the same amount of power? The answer is no. The environment acts back on the dipole and influences its ability to radiate.

According to Poynting's theorem (cf. Equation 5.6) the radiated power  $\bar{P}$  of any current distribution with a harmonic time dependence has to be identical to the rate

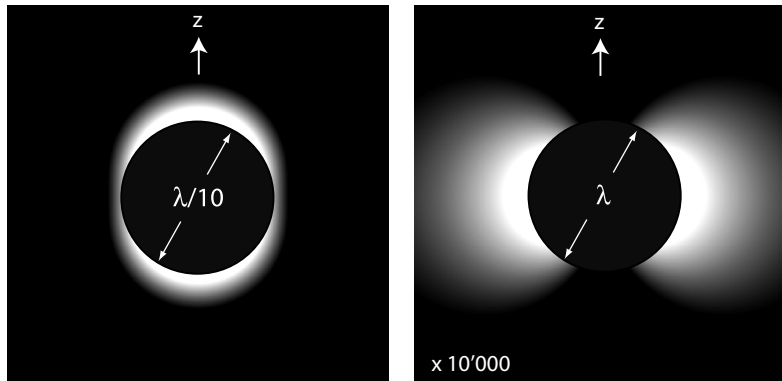


Figure 6.5: Electric energy density outside a fictitious sphere enclosing a dipole  $\mathbf{p} = p\mathbf{n}_z$ . (Left) Close to the dipole's origin the field distribution is elongated along the dipole axis (near-field). (Right) At larger distances the field spreads transversely to the dipole axis (far-field).

of energy dissipation  $dW/dt$  given by

$$\frac{dW}{dt} = -\frac{1}{2} \int_V \text{Re}\{\underline{\mathbf{j}}_0^* \cdot \underline{\mathbf{E}}\} dV, \quad (6.51)$$

$V$  being the source volume. The current density  $\underline{\mathbf{j}}_0$  is either a source current that generates the fields, or a loss current that is associated with thermal losses. Either way,  $\underline{\mathbf{j}}_0$  represents both energy sources and energy sinks. If we introduce the dipole's current density from Eq. (6.20) we obtain the important result

$$\bar{P} = \frac{\omega}{2} \text{Im} \{ \underline{\mathbf{p}}^* \cdot \underline{\mathbf{E}}(\mathbf{r}_0) \} \quad (6.52)$$

where the field  $\underline{\mathbf{E}}$  is evaluated at the dipole's origin  $\mathbf{r}_0$ . This equation can be rewritten in terms of the Green function by using Eq. (6.34) as

$$\bar{P} = \frac{\omega^3 |\underline{\mathbf{p}}|^2}{2c^2 \varepsilon_0 \varepsilon} \left[ \mathbf{n}_p \cdot \text{Im} \left\{ \underline{\underline{\mathbf{G}}}(\mathbf{r}_0, \mathbf{r}_0) \right\} \cdot \mathbf{n}_p \right], \quad (6.53)$$

with  $\mathbf{n}_p$  being the unit vector in the direction of the dipole moment.

At first sight it seems not possible to evaluate Eq. (6.52) since  $\exp(ikR)/R$  appears to be infinite at  $\mathbf{r} = \mathbf{r}_0$ . As we shall see this is not the case. We first note that due to the dot product between  $\underline{\mathbf{p}}$  and  $\underline{\mathbf{E}}$  we need only to evaluate the component of  $\underline{\mathbf{E}}$  in the direction of  $\underline{\mathbf{p}}$ . Choosing  $\underline{\mathbf{p}} = p \mathbf{n}_z$  we calculate  $\underline{E}_z$  as

$$\underline{E}_z = \frac{p}{4\pi \varepsilon_0 \varepsilon} \frac{e^{ikR}}{R} \left[ k^2 \sin^2 \vartheta + \frac{1}{R^2} (3 \cos^2 \vartheta - 1) - \frac{ik}{R} (3 \cos^2 \vartheta - 1) \right]. \quad (6.54)$$

Since the interesting part is the field at the origin of the dipole, the exponential term is expanded into a series [ $\exp(ikR) = 1 + ikR + (1/2)(ikR)^2 + (1/6)(ikR)^3 + \dots$ ] and the limiting case  $R \rightarrow 0$  is considered. Thus,

$$\frac{dW}{dt} = \lim_{R \rightarrow 0} \frac{\omega}{2} p \text{Im} \{ \underline{E}_z \} = \frac{\omega p^2}{8\pi \varepsilon_0 \varepsilon} \lim_{R \rightarrow 0} \left\{ \frac{2}{3} k^3 + R^2 (\dots) + \dots \right\} = \frac{p^2}{12\pi} \frac{\omega}{\varepsilon_0 \varepsilon} k^3, \quad (6.55)$$

which is identical with Eq. (6.49). Thus, Eq. (6.52) leads to the correct result despite the apparent singularity at  $R = 0$ .

The importance of Eq. (6.52) becomes obvious if we consider a dipole in an inhomogeneous environment, such as an antenna next to the earth surface. The



rate at which energy is released can still be calculated by integrating the Poynting vector over a surface enclosing the dipole. However, to do this, we need to know the electromagnetic field everywhere on the enclosing surface. Because of the inhomogeneous environment, this field is not equal to the dipole field alone! Instead, it is the self-consistent field, that is, the field  $\underline{\mathbf{E}}$  generated by the superposition of the dipole field and the scattered field from the environment (see Fig. 6.6). Thus, to determine the energy dissipated by the dipole we first need to determine the electromagnetic field everywhere on the enclosing surface. However, by using Eq. (6.52) we can do the same job by only evaluating the total field at the dipole's origin  $\mathbf{r}_0$ .

As illustrated in Fig. 6.6, we decompose the electric field at the dipole's position as

$$\underline{\mathbf{E}}(\mathbf{r}_0) = \underline{\mathbf{E}}_0(\mathbf{r}_0) + \underline{\mathbf{E}}_s(\mathbf{r}_0) , \quad (6.56)$$

where  $\underline{\mathbf{E}}_0$  and  $\underline{\mathbf{E}}_s$  are the primary dipole field and the scattered field, respectively. Introducing Eq. (6.56) into Eq. (6.52) allows us to split the rate of energy dissipation  $P = dW/dt$  into two parts. The contribution of  $\underline{\mathbf{E}}_0$  has been determined in Eqs. (6.49) and (6.55) as

$$\bar{P}_0 = \frac{p^2}{12\pi} \frac{\omega}{\epsilon_0 \epsilon} k^3 , \quad (6.57)$$

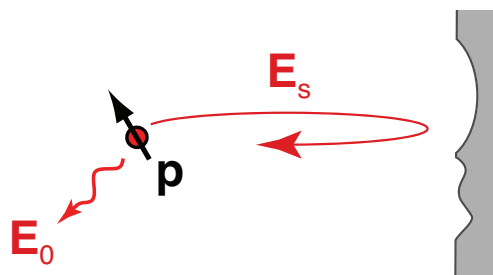


Figure 6.6: Illustration of dipole radiation in inhomogeneous environments. The total field is composed of a primary field  $\underline{\mathbf{E}}_0$  directly radiated by the dipole and a secondary field  $\underline{\mathbf{E}}_s$  that is emitted by the dipole and then scattered at inhomogeneities in the environment.

which allows us to write for the normalized rate of energy radiation

$$\frac{\bar{P}}{\bar{P}_0} = 1 + \frac{6\pi\epsilon_0\epsilon}{p^2} \frac{1}{k^3} \text{Im}\{\underline{\mathbf{p}}^* \cdot \underline{\mathbf{E}}_s(\mathbf{r}_0)\} \quad (6.58)$$

Thus, the change of energy dissipation depends on the *secondary field* of the dipole. This field corresponds to the dipole's own field emitted at a former time. It arrives at the position of the dipole after it has been scattered in the environment.

## 6.5 Fields Emitted by Arbitrary Sources

In Section 6.3.1 we have derived the vector potential  $\mathbf{A}$  of a time-harmonic dipole  $\mathbf{p}$ . Using the scalar free-space Green function  $G_0$  we have then found a solution for the vector potential of an arbitrary current distribution (see Eq. 6.33). The same procedure can be applied to the electric field vector  $\mathbf{E}$ .

According to Eq. (6.34) the  $\mathbf{E}$ -field can be expressed in terms of a dyadic (tensorial) Green function as  $\underline{\mathbf{E}}(\mathbf{r}) = \omega^2 \mu_0 \overset{\leftrightarrow}{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}') \underline{\mathbf{p}}$ , where  $\mathbf{r}'$  is the origin of the dipole. We can rewrite this equation as

$$\underline{\mathbf{E}}(\mathbf{r}) = \omega^2 \mu_0 \mu \int_V \overset{\leftrightarrow}{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}'') \underline{\mathbf{p}} \delta(\mathbf{r}' - \mathbf{r}'') dV'' \quad (6.59)$$

Using Eq. (6.20) for the current density of a dipole [ $\underline{\mathbf{j}}_0(\mathbf{r}'') = -i\omega \underline{\mathbf{p}} \delta(\mathbf{r}' - \mathbf{r}'')$ ] and substituting into Eq. (6.59) above, yields

$$\underline{\mathbf{E}}(\mathbf{r}) = i\omega \mu_0 \mu \int_V \overset{\leftrightarrow}{\mathbf{G}}_0(\mathbf{r}, \mathbf{r}') \underline{\mathbf{j}}_0(\mathbf{r}') dV' \quad (6.60)$$

where  $\underline{\mathbf{j}}_0$  now is an arbitrary current density distribution within the source volume  $V$ . We could have derived Eq. (6.60) also by following a more formal way using the definition of the Green function as described in Section 6.1. Fig. 6.7 illustrates the meaning of Eq. (6.60): The volume  $V$  is subdivided into infinitesimal units, each of which occupied by a point source with weight  $\underline{\mathbf{j}}_0(\mathbf{r}')$ . In a similar way we find the

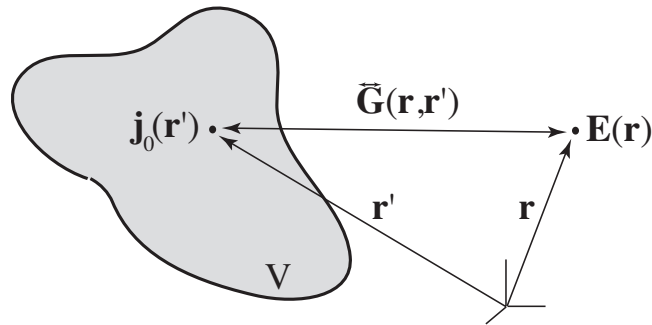


Figure 6.7: Illustration of the dyadic Green function  $\vec{\vec{G}}(\mathbf{r}, \mathbf{r}')$ . The Green function renders the electric field at the field point  $\mathbf{r}$  due to a single point source  $\mathbf{j}_0$  at the source point  $\mathbf{r}'$ . Since the field at  $\mathbf{r}$  depends on the orientation of  $\mathbf{j}$  the Green function must account for all possible orientations in the form of a tensor.

solution for the magnetic field as

$$\underline{\mathbf{H}}(\mathbf{r}) = \int_V [\nabla \times \vec{\vec{G}}_0(\mathbf{r}, \mathbf{r}')] \underline{\mathbf{j}}_0(\mathbf{r}') dV' \quad (6.61)$$

Note that mathematically, the fields  $\mathbf{E}$  and  $\mathbf{H}$  above are *particular* solutions of the differential equations (2.1) and (2.2). For a complete solution we need to superimpose the *homogeneous* solutions, which are solutions of (2.1) and (2.2) with the right sides being zero. These homogeneous solutions are fields that are present even in absence of the sources  $\underline{\mathbf{j}}_0$ .

## 6.6 Sources with Arbitrary Time-Dependence

So far we have considered the fields generated by a source that is oscillating harmonically in time with angular frequency  $\omega$ . But what if the time dependence is arbitrary, for example, a short pulse? In these cases we can employ Fourier transforms, which describe an arbitrary time dependence by a superposition of time harmonic dependences (see Section 2.2).

Let us go back to the time-harmonic solution (6.33) of the vector potential  $\mathbf{A}$ . We have pointed out in Section 2.2 that Maxwell's equations for the Fourier trans-

forms of the fields ( $\hat{\mathbf{E}}, \hat{\mathbf{H}}, \dots$ ) are formally the same as Maxwell's equations for the complex amplitudes ( $\underline{\mathbf{E}}(\mathbf{r}), \underline{\mathbf{H}}(\mathbf{r}), \dots$ ). Therefore, Eq. (6.33) implies that

$$\hat{\mathbf{A}}(\mathbf{r}, \omega) = \mu_0 \mu(\omega) \int_V \hat{G}_0(\mathbf{r}, \mathbf{r}', \omega) \hat{\mathbf{j}}_0(\mathbf{r}', \omega) dV' \quad (6.62)$$

where, according to Eq. (6.29),  $\hat{G}_0 = \exp(ik(\omega)|\mathbf{r} - \mathbf{r}'|) / (4\pi|\mathbf{r} - \mathbf{r}'|)$ , with  $k(\omega) = n(\omega)\omega/c$ .  $\hat{\mathbf{j}}_0$  denotes the Fourier transform of an arbitrary time-dependent current density, that is,

$$\hat{\mathbf{j}}_0(\mathbf{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{j}_0(\mathbf{r}, t) e^{i\omega t} dt. \quad (6.63)$$

The time-dependent vector potential  $\mathbf{A}(\mathbf{r}, t)$  of this current density is found by Fourier transforming Eq. (6.62), which yields

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{2\pi} \tilde{\mu}(t) * \int_V G_0(\mathbf{r}, \mathbf{r}', t) * \mathbf{j}_0(\mathbf{r}', t) dV', \quad (6.64)$$

where  $*$  denotes convolution in time and  $\tilde{\mu}(t)$  is the Fourier transform of  $\mu(\omega)$ .  $G_0(\mathbf{r}, \mathbf{r}', t)$  is given by

$$G_0(\mathbf{r}, \mathbf{r}', t) = \int_{-\infty}^{\infty} \hat{G}_0(\mathbf{r}, \mathbf{r}', \omega) e^{-i\omega t} d\omega. \quad (6.65)$$

Inserting the expression for  $\hat{G}_0$  yields

$$G_0(\mathbf{r}, \mathbf{r}', t) = \int_{-\infty}^{\infty} \frac{e^{ik(\omega)|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} e^{-i\omega t} d\omega = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \int_{-\infty}^{\infty} e^{-i\omega[t-n(\omega)|\mathbf{r}-\mathbf{r}'|/c]} d\omega. \quad (6.66)$$

In order to solve this integral we need to know the dependence of the index of refraction  $n$  on frequency  $\omega$ , which is referred to as *dispersion*.

### Dispersion-free Media

To simplify the discussion, we now assume that  $n(\omega) = n$  and  $\mu(\omega) = \mu$  (dispersion-free medium) and obtain<sup>3</sup>

$$G_0(\mathbf{r}, \mathbf{r}', t) = \frac{1}{2} \frac{\delta[t - |\mathbf{r} - \mathbf{r}'|n/c]}{|\mathbf{r} - \mathbf{r}'|}. \quad (6.67)$$

---

<sup>3</sup>  $\int \exp[ixy] dy = 2\pi\delta[x]$ .

Thus, the Green function in time domain is a simple delta function evaluated at the earlier time  $t' = t - nR/c$ , where  $t$  is the current time and  $R$  is the distance between source point  $\mathbf{r}'$  and observation point  $\mathbf{r}$ .

We now insert  $G_0(\mathbf{r}, \mathbf{r}', t)$  into Eq. (6.64) and obtain

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0 \mu}{4\pi} \int_V \int_{t'} \frac{\delta[t' - |\mathbf{r} - \mathbf{r}'|n/c]}{|\mathbf{r} - \mathbf{r}'|} \mathbf{j}_0(\mathbf{r}', t - t') dt' dV' \\ &= \frac{\mu_0 \mu}{4\pi} \int_V \frac{\mathbf{j}_0(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|n/c)}{|\mathbf{r} - \mathbf{r}'|} dV'. \end{aligned} \quad (6.68)$$

A similar equation can be derived for the scalar potential  $\phi(\mathbf{r})$ . Taken both together we have

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 \mu}{4\pi} \int_V \frac{\mathbf{j}_0(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|n/c)}{|\mathbf{r} - \mathbf{r}'|} dV' \quad (6.69)$$

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0\epsilon} \int_V \frac{\rho_0(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|n/c)}{|\mathbf{r} - \mathbf{r}'|} dV' \quad (6.70)$$

These equations state that the fields  $\mathbf{A}$  and  $\phi$  at the location  $\mathbf{r}$  and time  $t$  are determined by the sources  $\mathbf{j}_0$  and  $\rho_0$  at location  $\mathbf{r}'$  at the earlier time  $t - |\mathbf{r} - \mathbf{r}'|n/c$ . The earlier time is a consequence of the speed of light  $c$ : it takes a time of  $|\mathbf{r} - \mathbf{r}'|n/c$  for the fields to travel a distance of  $|\mathbf{r} - \mathbf{r}'|$  in a medium with index of refraction  $n$ . Thus, Maxwell's equations explain the mysterious "action-at-distance" phenomenon discussed in the introduction of this course (see Fig. 1). It has to be emphasized that the index of refraction  $n$  and hence the material parameters  $\mu$  and  $\epsilon$  are assumed to be dispersion-free, which is an approximation. The only material that is truly dispersion-free is vacuum ( $n = 1$ ).

To find the fields  $\mathbf{E}$  and  $\mathbf{H}$  we insert the solutions of  $\mathbf{A}$  and  $\phi$  into Eqs. (6.7) and (6.8). The calculation is not straightforward because  $\mathbf{A}$  and  $\phi$  depend on the retarded time  $t - |\mathbf{r} - \mathbf{r}'|n/c$ . Therefore, an operation on the spatial coordinates (e.g.  $\nabla \times$ ) is implicitly also an operation on time. We will not go through this exercise and only mention that the solution is identical with Eq. (2) if we express  $\rho$  and  $\mathbf{j}$  with the charge and current densities of a discrete charge (c.f. Equation 3 and 4). The result has three terms: the first term depends on the charge, the second term on the velocity of the charge, and the third term on the acceleration of charge. It is

the latter that is associated with electromagnetic radiation.

The expression of fields in terms of retarded time is of limited practical value. They help us to understand the physical origin of radiation but carrying out the integrals in Eqs. (6.69) and (6.70) is nearly impossible for realistic sources. Furthermore, the time-domain approach taken here is not able to accommodate dispersive materials. Therefore, it is generally more favorable to process the fields in Fourier space, that is, first calculate the spectrum of the source via Fourier transformation, then calculate the spectra of the fields, and finally taking the inverse Fourier transform to express the fields in time domain. This procedure is shown in Fig. 6.8.

### 6.6.1 Dipole Fields in Time Domain

We have calculated the fields of a dipole with harmonic time dependence in Section 6.3.2. These fields were expressed in spherical vector components (see Eq. 6.44 - 6.46). Note that these fields are the complex amplitudes and that the time-dependent fields are arrived at by multiplying with  $\exp[-i\omega t]$  and taking the real part.

Remember, that Maxwell's equations for the complex amplitudes of time harmonic fields (Eq. 2.31–2.34) are identical with Maxwell's equations for the Fourier transforms of fields with arbitrary time dependence (Eq. 2.25–2.28). Therefore, the solutions are identical as well. For example, the spectrum of the dipole's  $E_\theta$

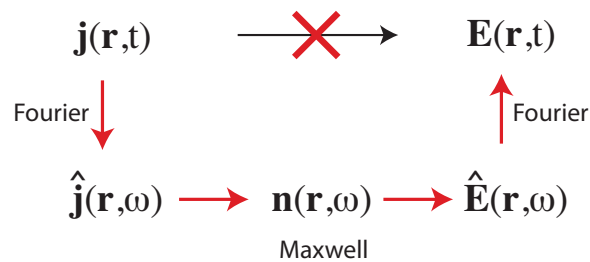


Figure 6.8: Calculating the field  $\mathbf{E}$  of a time-dependent source  $\mathbf{j}$ . Because of dispersion and retardation it is favorable to solve for the fields in frequency space.

field is

$$\hat{E}_\vartheta(\mathbf{r}, \omega) = -\hat{p}(\omega) \frac{\sin \vartheta}{4\pi\epsilon_0\epsilon(\omega)} \frac{\exp[ik(\omega)r]}{r} k^2(\omega) \left[ \frac{1}{k^2(\omega)r^2} - \frac{i}{k(\omega)r} - 1 \right], \quad (6.71)$$

in analogy to the corresponding complex amplitude in Eq. (6.45). We used  $\hat{p}(\omega) = |\hat{\mathbf{p}}(\omega)|$ . Note that dispersion is fully accounted for through  $\epsilon(\omega)$  and  $k(\omega)$ . Using the time-dependent field  $E_\vartheta(\mathbf{r}, t)$  can simply be calculated using the Fourier transform (2.23).

To illustrate the transformation from frequency to time domain, we assume that the dipole is in vacuum ( $k = \omega/c$  and  $\epsilon = 1$ ). Also, we will only consider the far-field term in Eq. (6.71). The time-dependent far-field  $E_\vartheta^f$  is calculated as

$$E_\vartheta^f(\mathbf{r}, t) = \int_{-\infty}^{\infty} \hat{E}_\vartheta^f(\mathbf{r}, \omega) e^{-i\omega t} d\omega = \frac{\sin \vartheta}{4\pi\epsilon_0} \frac{1}{c^2 r} \int_{-\infty}^{\infty} \omega^2 \hat{p}(\omega) e^{-i\omega(t-r/c)} d\omega. \quad (6.72)$$

To solve this integral we set  $\omega^2 \hat{p}(\omega) = \hat{\psi}(\omega)$ , where the Fourier transform of  $\hat{\psi}(\omega)$  is  $\psi(t)$ . Using the translation property of Fourier-transforms we obtain

$$\int_{-\infty}^{\infty} \hat{\psi}(\omega) e^{-i\omega(t-r/c)} d\omega = \psi(t - r/c). \quad (6.73)$$

Thus, it remains to solve for  $\psi(t)$ :

$$\psi(t) = \int_{-\infty}^{\infty} \omega^2 \hat{p}(\omega) e^{-i\omega t} d\omega = -\frac{d^2}{dt^2} p(t). \quad (6.74)$$

Putting the pieces together we finally find

$$E_\vartheta^f(\mathbf{r}, t) = -\frac{\sin \vartheta}{4\pi\epsilon_0} \frac{1}{c^2 r} \left. \frac{d^2 p(\tau)}{d\tau^2} \right|_{\tau=t-r/c} \quad (6.75)$$

Thus, the field at  $\mathbf{r} = [r, \vartheta, \varphi]$  and time  $t$  is determined by the dipole at  $\mathbf{r}' = 0$  at the earlier time  $t - r/c$ . As before, we find that it takes a time  $r/c$  for the “action” to travel from the dipole to the observation point. Other terms of the dipole fields (Eqs. 6.44–6.46) can be calculated following the same procedure. The result is

$$E_r(t) = \frac{\cos \vartheta}{4\pi\epsilon_0} \left[ \frac{2}{r^3} + \frac{2}{c r^2} \frac{d}{d\tau} \right] p(\tau) \Big|_{\tau=t-r/c} \quad (6.76)$$

$$E_\vartheta(t) = -\frac{\sin \vartheta}{4\pi\epsilon_0} \left[ \frac{1}{r^3} + \frac{1}{c r^2} \frac{d}{d\tau} + \frac{1}{c^2 r} \frac{d^2}{d\tau^2} \right] p(\tau) \Big|_{\tau=t-r/c} \quad (6.77)$$

$$H_\varphi(t) = -\frac{\sin \vartheta}{4\pi\epsilon_0} \sqrt{\frac{\epsilon_0}{\mu_0}} \left[ \frac{1}{c r^2} \frac{d}{d\tau} + \frac{1}{c^2 r} \frac{d^2}{d\tau^2} \right] p(\tau) \Big|_{\tau=t-r/c} \quad (6.78)$$

We see that the far-field is generated by the acceleration of the charges that constitute the dipole moment. Similarly, the intermediate-field and the near-field are generated by the speed and the position of the charges, respectively.

## 6.7 The Lorentzian Power Spectrum

The spectrum of various physical processes is characterized by narrow lines described by Lorentzian line shape functions. Examples are the spontaneous emission by atoms or molecules, laser radiation, or microwave resonators. To understand the origin of Lorentzian line shapes we consider a dipole located at  $\mathbf{r}_0 = 0$  that starts to oscillate at time  $t = 0$ . The observer is assumed to be at large distance from the dipole, which allows us to restrict the discussion to the dipole's far-field  $E_{\vartheta}^f(\mathbf{r}, t)$ .

The equation of motion for an undriven harmonically oscillating dipole is

$$\frac{d^2}{dt^2}\mathbf{p}(t) + \gamma_0 \frac{d}{dt}\mathbf{p}(t) + \omega_0^2\mathbf{p}(t) = 0. \quad (6.79)$$

The natural frequency of the oscillator is  $\omega_0$  and its damping constant is  $\gamma_0$ . The solution for  $\mathbf{p}$  is

$$\mathbf{p}(t) = \text{Re} \left\{ \mathbf{p}_0 e^{-i\omega_0 \sqrt{1 - (\gamma_0^2/4\omega_0^2)} t} e^{-\gamma_0 t/2} \right\}. \quad (6.80)$$

Typically, the damping constant is much smaller than the oscillation frequency ( $\gamma_0 \ll \omega_0$ ), which implies  $\sqrt{1 - (\gamma_0^2/4\omega_0^2)} \approx 1$ .

The spectrum  $\hat{E}_{\vartheta}(\omega)$  detected by the observer is (cf. Eq. (2.24))

$$\hat{E}_{\vartheta}(\omega) = \frac{1}{2\pi} \int_{r/c}^{\infty} E_{\vartheta}(t) e^{i\omega t} dt. \quad (6.81)$$

Here we set the lower integration limit to  $t = r/c$  because the dipole starts emitting at  $t = 0$  and it takes the time  $t = r/c$  for the radiation to propagate to the observation point. Therefore  $E_{\vartheta}(t < r/c) = 0$ . Inserting the solution for the dipole moment from Eq. (6.80) and making use of  $\gamma_0 \ll \omega_0$  we obtain after integration

$$\hat{E}_{\vartheta}(\omega) = \frac{1}{2\pi} \frac{p_0 \sin\vartheta \omega_0^2}{8\pi\epsilon_0 c^2 r} \left[ \frac{\exp(i\omega r/c)}{i(\omega + \omega_0) - \gamma_0/2} + \frac{\exp(i\omega r/c)}{i(\omega - \omega_0) - \gamma_0/2} \right]. \quad (6.82)$$



The energy radiated into the unit solid angle  $d\Omega = \sin\vartheta d\vartheta d\varphi$  is calculated as

$$\frac{dW}{d\Omega} = \int_{-\infty}^{\infty} I(\mathbf{r}, t) r^2 dt = r^2 \sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{-\infty}^{\infty} |E_{\vartheta}(t)|^2 dt = 2\pi r^2 \sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{-\infty}^{\infty} |\hat{E}_{\vartheta}(\omega)|^2 d\omega, \quad (6.83)$$

where we applied Parseval's theorem and used the definition of the intensity  $I = \sqrt{\varepsilon_0/\mu_0} |E_{\vartheta}|^2$  of the emitted radiation. The total energy per unit solid angle  $d\Omega$  and per unit frequency interval  $d\omega$  can now be expressed as

$$\frac{dW}{d\Omega d\omega} = \frac{1}{4\pi\varepsilon_0} \frac{p_0^2 \sin^2\vartheta \omega_0^4}{4\pi^2 c^3 \gamma_0^2} \left[ \frac{\gamma_0^2/4}{(\omega - \omega_0)^2 + \gamma_0^2/4} \right] \quad (6.84)$$

The spectral shape of this function is determined by the expression in the brackets known as the *Lorentzian lineshape function*. The function is shown in Fig. 6.9. The width of the curve measured at half its maximum height is  $\Delta\omega = \gamma_0$ , and is called "radiative linewidth."

Integrating the lineshape function over the entire spectral range yields a value of  $\pi\gamma_0/2$ . Integrating Eq. (6.84) over all frequencies and all directions leads to the totally radiated energy

$$W = \frac{p_0^2}{4\pi\varepsilon_0} \frac{\omega_0^4}{3c^3 \gamma_0}. \quad (6.85)$$

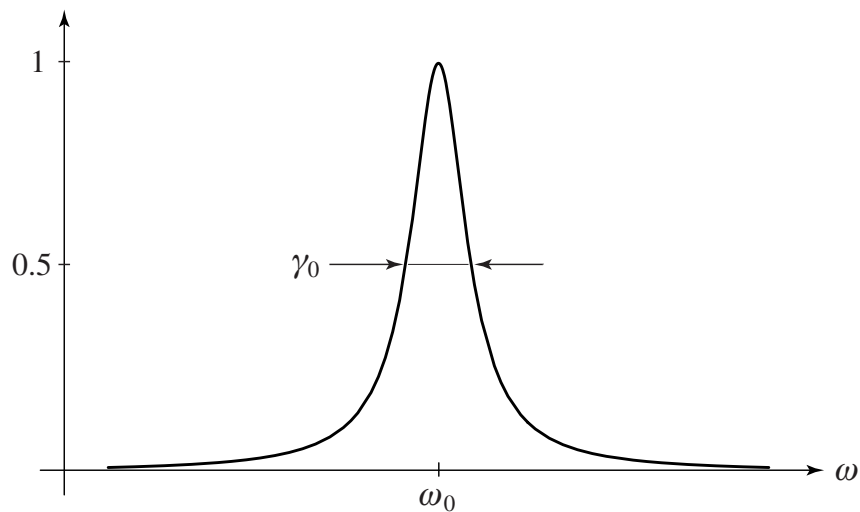


Figure 6.9: Lorentzian lineshape function as defined by the expression in brackets in Eq. (6.84).

This value is equal to the average power  $\bar{P}$  radiated by a driven harmonic oscillator divided by the linewidth  $\gamma_0$  (cf. Equation 6.49).