Lecture Notes on

ELECTROMAGNETIC FIELDS AND WAVES

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Introduction

The properties of electromagnetic fields and waves are most commonly discussed in terms of the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic induction field $\mathbf{B}(\mathbf{r}, t)$. The vector $\mathbf{r}$ denotes the location in space where the fields are evaluated. Similarly, t is the time at which the fields are evaluated. Note that the choice of $\mathbf{E}$ and $\mathbf{B}$ is arbitrary and that one could also proceed with combinations of the two, for example, with the vector and scalar potentials $\mathbf{A}$ and ϕ, respectively.

The fields $\mathbf{E}$ and $\mathbf{B}$ have been originally introduced to escape the dilemma of "action-at-distance', that is, the question of how forces are transferred between two separate locations in space. To illustrate this, consider the situation depicted in Figure 1. If we shake a charge at $\mathbf{r}_1$ then a charge at location $\mathbf{r}_2$ will respond. But how did this action travel from $\mathbf{r}_1$ to $\mathbf{r}_2$? Various explanations were developed over the years, for example, by postulating an aether that fills all space and that acts as a transport medium, similar to water waves. The fields $\mathbf{E}$ and $\mathbf{B}$ are pure constructs to deal with the "action-at-distance' problem. Thus, forces generated by

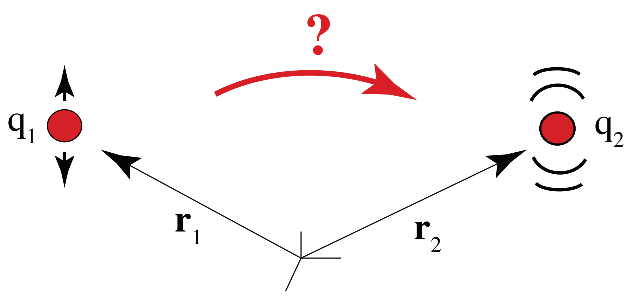

Figure 1: Illustration of "action-at-distance". Shaking a charge at $\mathbf{r}_1$ makes a second charge at $\mathbf{r}_2$ respond.

electrical charges and currents are explained in terms of $\mathbf{E}$ and $\mathbf{B}$, quantities that we cannot measure directly.

Basic Properties

As mentioned above, $\mathbf{E}$ and $\mathbf{B}$ have been introduced to explain forces acting on charges and currents. The Coulomb force (electric force) is mediated by the electric field and acts on the charge q, that is, $\mathbf{F}_{\mathrm{e}} = q\mathbf{E}$. It accounts for the attraction or repulsion between static charges. The interaction of static charges is referred to as *electrostatics*. On the other hand, the Lorentz force (magnetic force) accounts for the interaction between static currents (charges traveling at constant velocities $\mathbf{v} = \dot{\mathbf{r}}$) according to $\mathbf{F}_{\mathrm{m}} = q\mathbf{v} \times \mathbf{B}$. The interaction of static currents is referred to as *magnetostatics*. Taken the electric and magnetic forces together we arrive at

$$\mathbf{F}(\mathbf{r},t) = q\,[\mathbf{E}(\mathbf{r},t)\; + \;\mathbf{v}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t)] \tag{1}$$

In the SI unit system, force is measured in Newtons (N = J / m = A V s / m) and charge in terms of Coulomb (C = A s). Equation (1) therefore imposes the following units on the fields: [E] = V/m and [B] = V s / m^2. The latter is also referred to as Tesla (T).

Interestingly, the fields $\mathbf{E}$ and $\mathbf{B}$ depend on the observer's reference frame. In fact, the field $\mathbf{E}$ in one inertial frame can be equal to the field $\mathbf{B}$ in another inertial

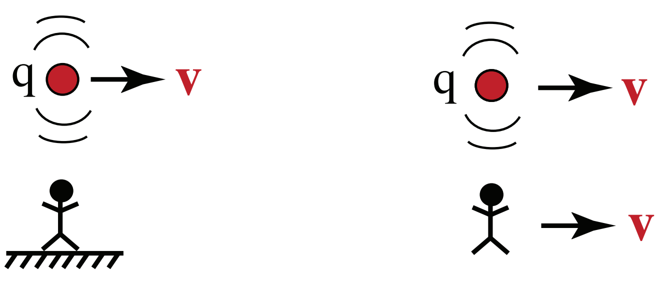

Figure 2: The fields $\mathbf{E}$ and $\mathbf{B}$ depend on the inertial frame. An observer at rest sees a $\mathbf{B}$ field when a charge at velocity $\mathbf{v}$ moves by (left). However, an observer moving at the same speed will experience only an $\mathbf{E}$ field.

frame. To illustrate this consider the two situations shown in Figure 2 where an observer is measuring the fields of a charge moving at velocity $\mathbf{v}$. An observer at rest will measure a $\mathbf{B}$ field whereas an observer moving at the same speed as the charge will only experience an $\mathbf{E}$ field. Why? Because the charge appears to be at rest from the observer's point of view.

In general, the electric field measured by an observer at $\mathbf{r}_\mathrm{o}$ and at time t can be expressed as (see R. Feynman 'Lectures on Physics', Vol. II, p 21-1)

$$\mathbf{E}(\mathbf{r}_\mathrm{o}, t) \;=\; \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{n}_{\mathrm{r}'}}{\mathrm{r}'^2} + \frac{\mathrm{r}'}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{n}_{\mathrm{r}'}}{\mathrm{r}'^2}\right) + \frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{n}_{\mathrm{r}'}\right] , \qquad (2)$$

where $c = 2.99792456\,..\,10^8$ m/s is the speed of light. As shown in Figure 3, r' is the distance between the charge and the observer at the earlier time $(t - \mathrm{r}'/c)$. Similarly, $\mathbf{n}_{\mathrm{r}'}$ is the unit vector pointing from the charge towards the observer at the earlier time $(t - \mathrm{r}'/c)$. Thus, the field at the observation point $\mathbf{r}_\mathrm{o}$ at the time t depends on the motion of the charge at the earlier time $(t - \mathrm{r}'/c)$! The reason is that it takes a time $\Delta t = \mathrm{r}'/c$ for the field to travel the distance r' to the observer.

Let us discuss the different terms in Eq. (2). The first term is proportional to the *position* of the charge and describes a retarded Coulomb field. The second term is proportional to the *velocity* of the charge. Together with the first term it describes the instantaneous Coulomb field. The third term is proportional to the *acceleration* of the charge and is associated with electromagnetic radiation.

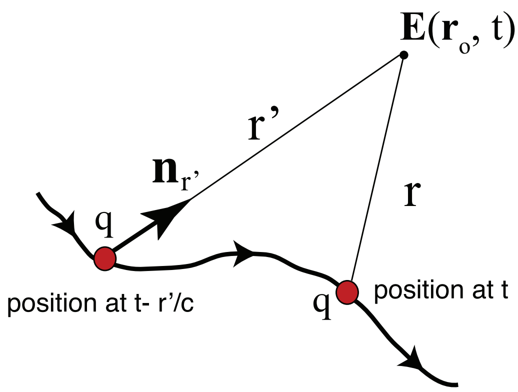

Figure 3: The field at the observation point $\mathbf{r}_\mathrm{o}$ at the time t depends on the motion of the charge at the earlier time $(t - \mathrm{r}'/c)$.

The objective of this course is to establish the theoretical foundations that lead to Eq. (2) and to develop an understanding for the generation and propagation of electromagnetic fields.

Microscopic and Macroscopic Electromagnetism

In microscopic electromagnetism one deals with discrete point charges $\mathrm{q_n}$ located at $\mathbf{r}_\mathrm{n}$ (see Figure 4). The charge density ρ and the current density $\mathbf{j}$ are then expressed as sums over Dirac delta functions

$$\rho(\mathbf{r}) \;=\; \sum_\mathrm{n} \mathrm{q_n}\, \delta[\mathbf{r}-\mathbf{r}_\mathrm{n}]\;, \tag{3}$$

$$\mathbf{j}(\mathbf{r}) \;=\; \sum_\mathrm{n} \mathrm{q_n}\, \dot{\mathbf{r}}_\mathrm{n}\, \delta[\mathbf{r}-\mathbf{r}_\mathrm{n}]\;, \tag{4}$$

where $\mathbf{r}_\mathrm{n}$ denotes the position vector of the nth charge and $\dot{\mathbf{r}}_\mathrm{n} = \mathbf{v}_\mathrm{n}$ its velocity. The total charge and current of the particle are obtained by a volume integration over ρ and $\mathbf{j}$. In terms of ρ and $\mathbf{j}$, the force law in Eq.(1) can be written as

$$\mathbf{F}(\mathbf{r},t) = \int_V \left[\rho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t) \;+\; \mathbf{j}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)\right] \mathrm{d}V \tag{5}$$

where V is the volume that contains all the charges q_n.

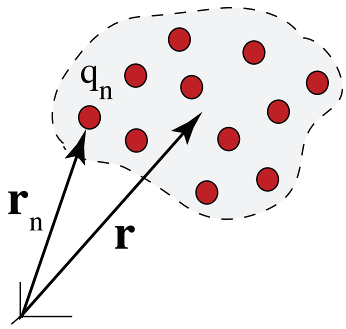

Figure 4: In the microscopic picture, optical radiation interacts with the discrete charges $\mathrm{q_n}$ of matter.

In macroscopic electromagnetism, ρ and $\mathbf{j}$ are viewed as continuous functions of position. Thus, the microscopic structure of matter is not considered and the fields become local spatial averages over microscopic fields. This is similar to the flow of water, for which the atomic scale is irrelevant.

In this course we will predominantly consider macroscopic fields for which ρ and $\mathbf{j}$ are smooth functions in space. However, the discrete nature can always be recovered by substituting Eqs. (3) and (4).

Pre-Maxwellian Electrodynamics

Let us review and summarize the laws of Ampère (Oersted), Faraday and Gauss, as introduced in the courses *Netzwerke und Schaltungen I & II*.

$$\begin{aligned}
\int_{\partial V} \mathbf{E}(\mathbf{r},t)\cdot\mathbf{n}\,da &= \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{r},t)\,dV && \text{Gauss' law (Cavendish 1772)}\\
\int_{\partial A} \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{s} &= -\frac{\partial}{\partial t}\int_A \mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,da && \text{Faraday's law (Faraday 1825)}\\
\int_{\partial A} \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{s} &= \mu_0\int_A \mathbf{j}(\mathbf{r},t)\cdot\mathbf{n}\,da && \text{Ampere's law (Oersted 1819)}\\
\int_{\partial V} \mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,da &= 0 && \text{No magnetic monopoles}
\end{aligned} \tag{6}$$

In our notation, V is a volume composed of infinitesimal volume elements dV, A is a surface composed of infinitesimal surface elements da, and $d\mathbf{s}$ is an infinitesimal line element. ∂V denotes the closed surface of the volume V. Similarly, ∂A is the circumference of the area A. $\mathbf{n}$ is a unit vector normal to the surface ∂V or circumference ∂A. The constants appearing in Eq. (6) are defined as follows

$$\mu_0 = 4\pi\, 10^{-7}\, \frac{\text{Vs}}{\text{Am}} = 1.2566370\, 10^{-6}\, \frac{\text{Vs}}{\text{Am}} \quad \text{(magnetic permeability)}$$

$$\varepsilon_0 = \frac{1}{\mu_0\, c^2} = 8.8541878\, 10^{-12}\, \frac{\text{As}}{\text{Vm}} \quad \text{(electric permittivity)}$$

where $c = 2.99792458\, 10^8$ m/s is the vacuum speed of light. Figure 5 illustrates the meaning of the four equations (6). The first equation, Gauss' law, states that the flux of electric field through a closed surface ∂V is equal to the total charge q inside ∂V. The second equation, Faraday's law, predicts that the electric field integrated along a loop ∂A corresponds to the time rate of change of the magnetic flux through the loop. Similarly, Ampère's law, states that the magnetic field integrated along a loop ∂A is equal to the current flowing through the loop. Finally, the fourth equation states that the flux of magnetic field through a closed surface is always zero which, in an analogy to Gauss' law, indicates that there are no magnetic charges.

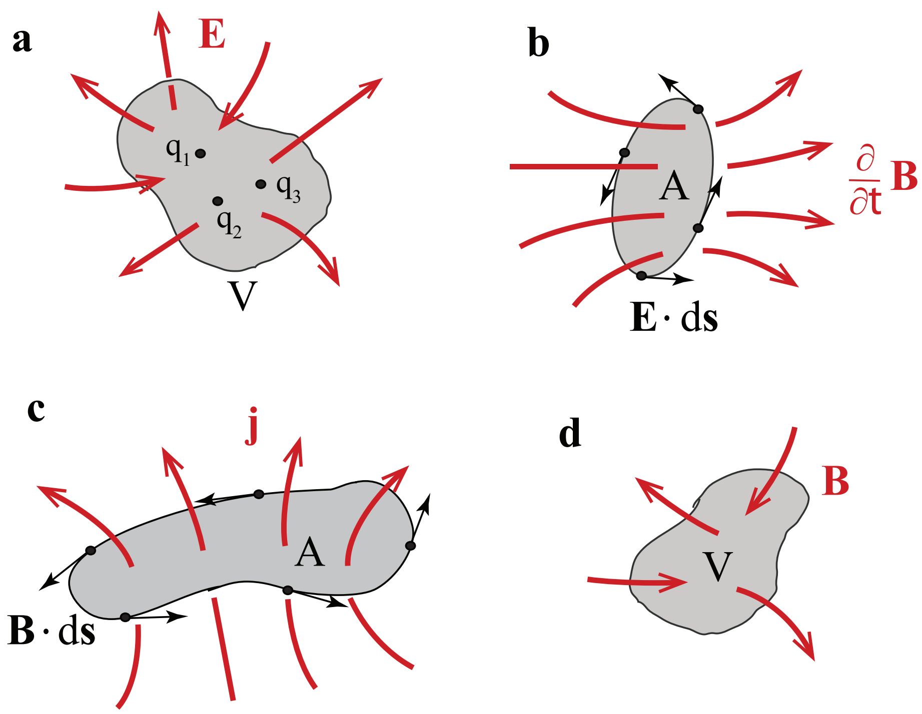

Figure 5: Illustration of (a) Gauss' law, (b) Faraday's law, (c) Ampere's law, and (d) the non-existence of magnetic charges.

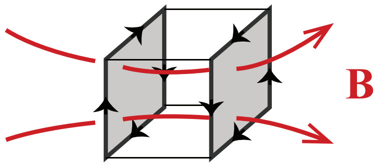

Figure 6: Ampère's law applied to a cube. Opposite faces cancel the magnetic circulation $\int_{\partial A} \mathbf{B}(\mathbf{r},t) \cdot d\mathbf{s}$, predicting that the flux of current through any closed surface is zero.

Let us consider Ampère's law for the different end faces of a small cube (c.f. Fig. 6). [1] It turns out that the magnetic field integrated along the circumference of one end face ($\int_{\partial A} \mathbf{B}(\mathbf{r},t) \cdot d\mathbf{s}$) is just the negative of the magnetic field integrated along the circumference of the opposite end face. Thus, the combined flux (seen from the outside) is zero. The same is true for the other pairs of end faces. Therefore, for any closed surface Ampère's law reduces to

$$\int_{\partial V} \mathbf{j}(\mathbf{r},t) \cdot \mathbf{n}\, da = 0 \quad \text{Kirchhoff I}. \tag{7}$$

In other words, the flux of current through any closed surface is zero: What flows in has to flow out.

Eq. (7) defines the familiar current law of Kirchhoff (Knotenregel). On the other hand, Kirchhoff's voltage law (Maschenregel) follows from Faraday's law if no time-varying magnetic fields are present. In this case,

$$\int_{\partial A} \mathbf{E}(\mathbf{r},t) \cdot d\mathbf{s} = 0 \quad \text{Kirchhoff II}. \tag{8}$$

The two Kirchhoff laws form the basis for circuit theory and electronic design.

[1] An arbitrary volume can be viewed as a sum over infinitesimal cubes.